Bmad Training Workshop

Content 07/29/2024

1. Introduction – background and goals of participants
2. The tutorial that we will cover – designing an ESR
3. Linear optics – Twiss parameters
4. FODO cells
5. Dispersion matching

Content 07/30/2024

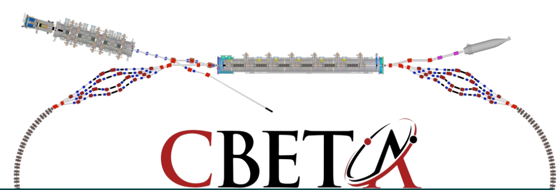
1. Twiss matching
2. Coordinate systems in Bmad
3. Combining arcs and straights to a ring
4. Low beta insertions
5. Different uses of Twiss parameters

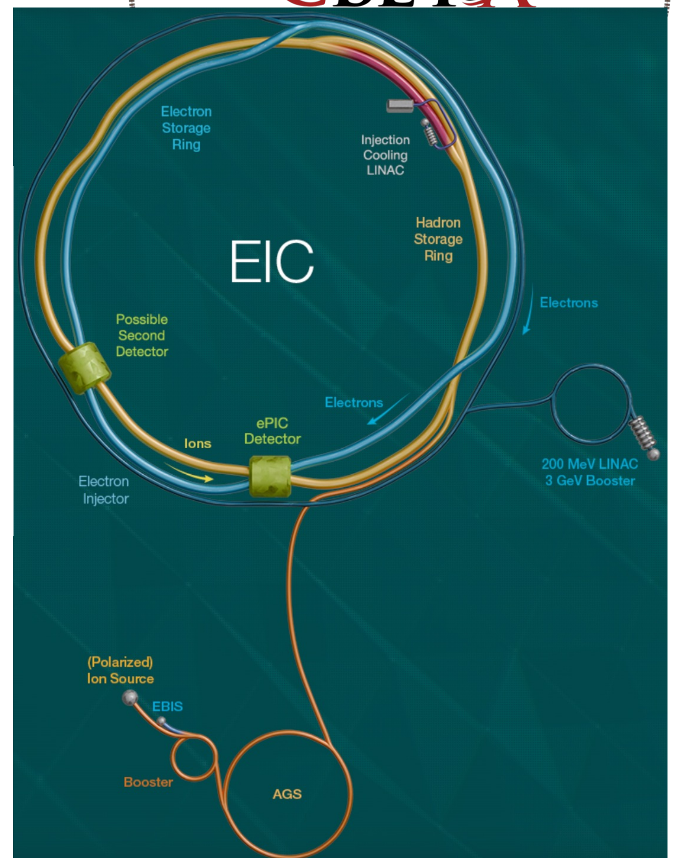
Content 07/31/2024

1. Tune-control cells
2. RF systems in a ring and long-term tracking
3. Programing with the Bmad library

Content 08/30/2024

1. Energy dependent Twiss
2. Sextupoles / Chromaticity
3. Chromatic beta beat
4. Dynamic aperture





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Class data

Dates: Monday – Friday July 29 – Augus 2, 2024

9am to 12:15 and 1:15 to 4:30pm, Friday to 12:15pm

Expectation: Collaborate on exercises.

Attend regularly to consult TAs for exercises on optics design.

Co-instructor for Bmad, design, and simulation: David Sagan dcs16@cornell.edu

TAs: Ningdong Wang nw285@cornell.edu, Matthew Signorelli mgs255@cornell.edu

Jonathan Unger jeu8@cornell.edu, Eiad Hamwi eh652@cornell.edu, Daria Kuzovkova dk638@cornell.edu

BNL help: Scott Berg jsberg@bnl.gov

Discuss with these and with each other at the Bmad SLACK channel **bmad-simulation.slack.com**

Lecture notes, the tutorial, and example solutions can all be accessed through

https://indico.classe.cornell.edu/event/2433/

Images

Images are taken from many sources, including:

- The Physics of Particle Accelerators, Klaus Wille, Oxford University Press, 2000, ISBN: 19 850549 3
- Particle Accelerator Physics I, Helmut Wiedemann, Springer, 2nd edition, 1999, ISBN 3 540 64671 x
- Teilchenbeschleuniger und Ionenoptic, Frank Hinterberger, 1997, Springer, ISBN 3 540 61238 6
- Introduction to Ultraviolet and X-ray Free-Electron Lasers, Martin Dohlus, Peter Schmüser, Jörg Rossbach, Springer, 2008
- Various web pages, 2003 – 2024

Literature

- Ring Design Tutorial https://www.classe.cornell.edu/bmad/tutorial_ring_design.pdf
- Bmad manual https://www.classe.cornell.edu/bmad/bmad-manual-2024-07-14.pdf

Introduction

The Physics of Particle Accelerators: An Introduction, Klaus Wille, Oxford University Press

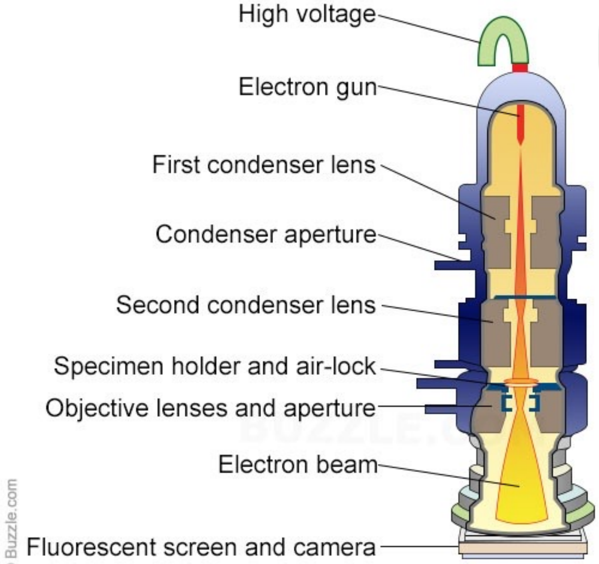
Wide selection of well explained topics

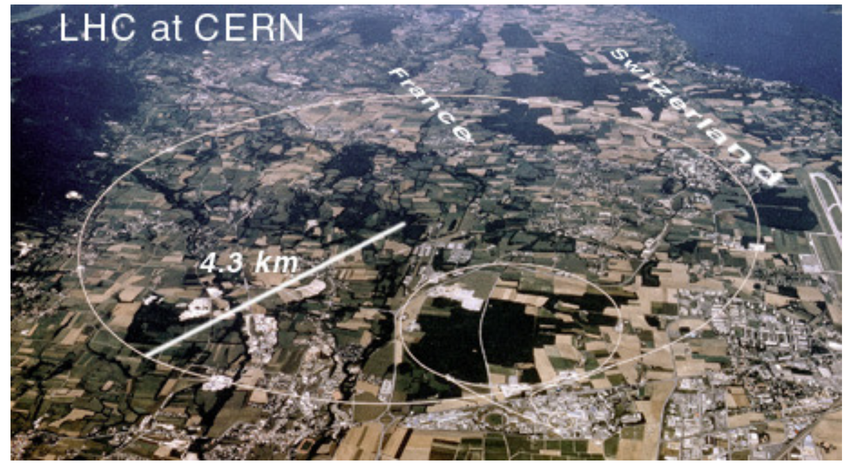
Particle Accelerator Physics, Helmut Wiedemann, Springer, (preferably 3rd edition)

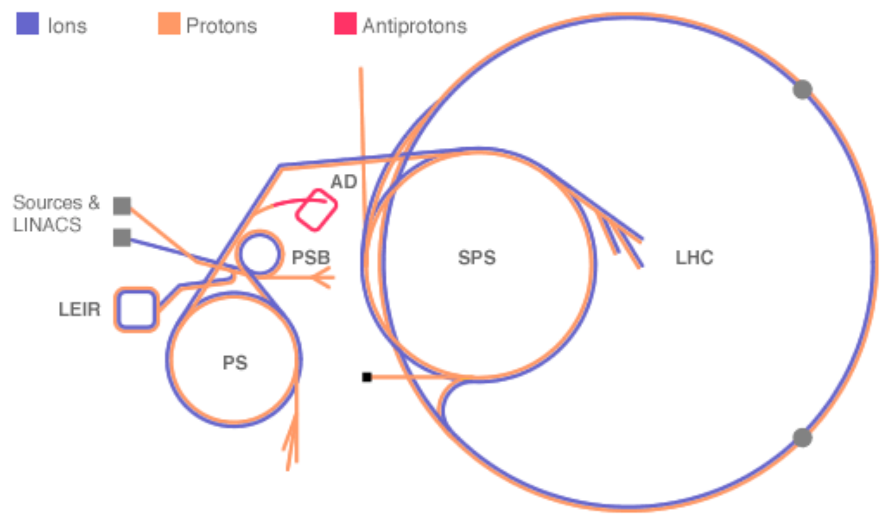
Tremendous overview, with references for derivations and explanations

Handbook of Accelerator Physics and Engineering, Alexander Wu Cao, Maury Tigner, Hans Weise, Frank Zimmermann (3rd edition)

Bmad for large and small particle accelerators







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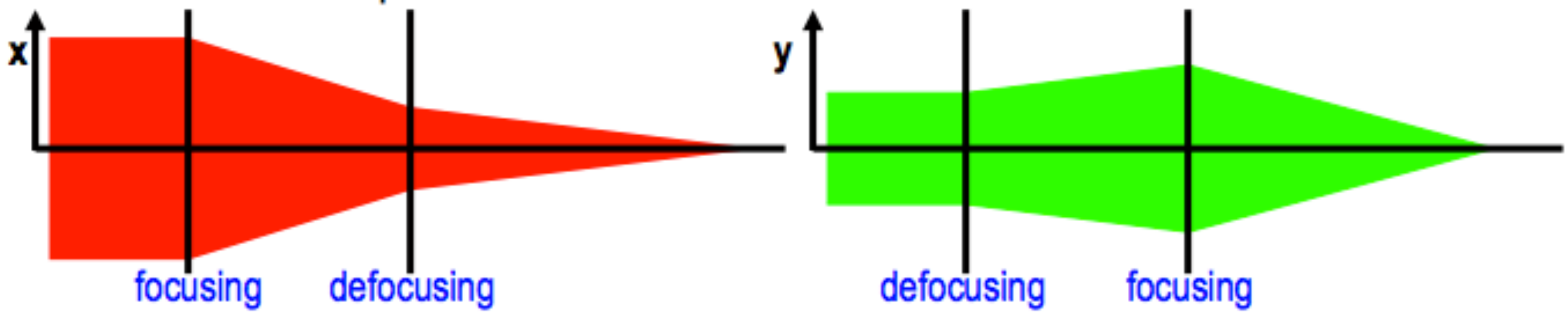
Today's focus: Quadrupole Focusing

Transverse fields defocus in one plane if they focus in the other plane.
But two successive elements, one focusing the other defocusing,
can focus in both planes:



Goal of the Ring Design Tutorial

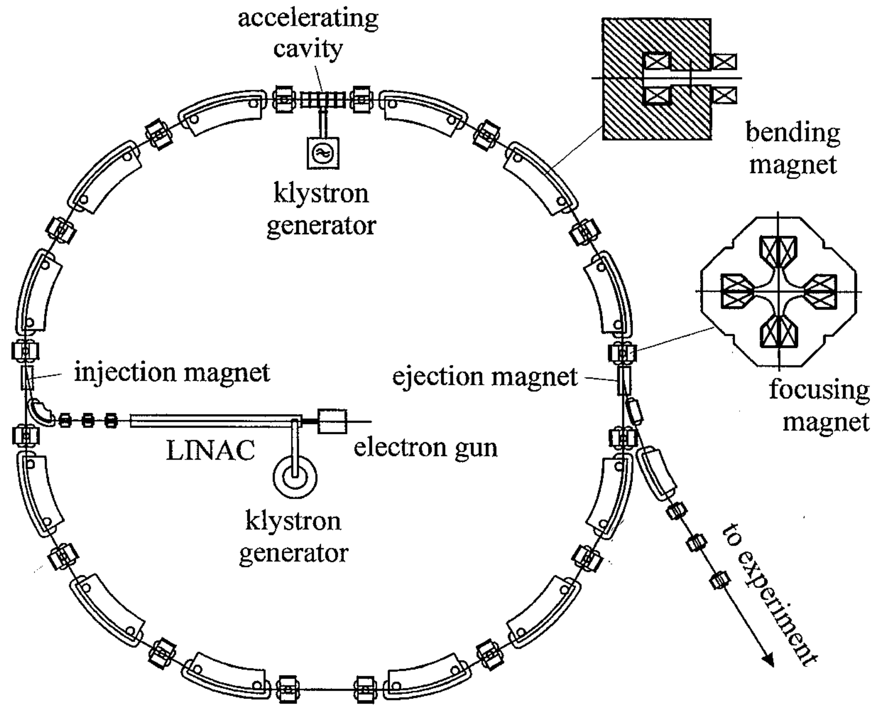
- 1945: Veksler (UDSSR) and McMillan (USA) invent the synchrotron
- 1946: Goward and Barnes build the first synchrotron (using a betatron magnet)
- 1949: Wilson et al. at Cornell are first to store beam in a synchrotron
  (later 300MeV, magnet of 80 Tons)
- 1949: McMillan builds a 320MeV electron synchrotron

- Many smaller magnets instead of one large magnet
- Only one acceleration section is needed, with

$$R = \frac{p(t)}{qB(R,t)} = \text{const.}$$

$$\omega = 2\pi \frac{v_{\text{particle}}}{L} n$$

for an integer n called the harmonic number



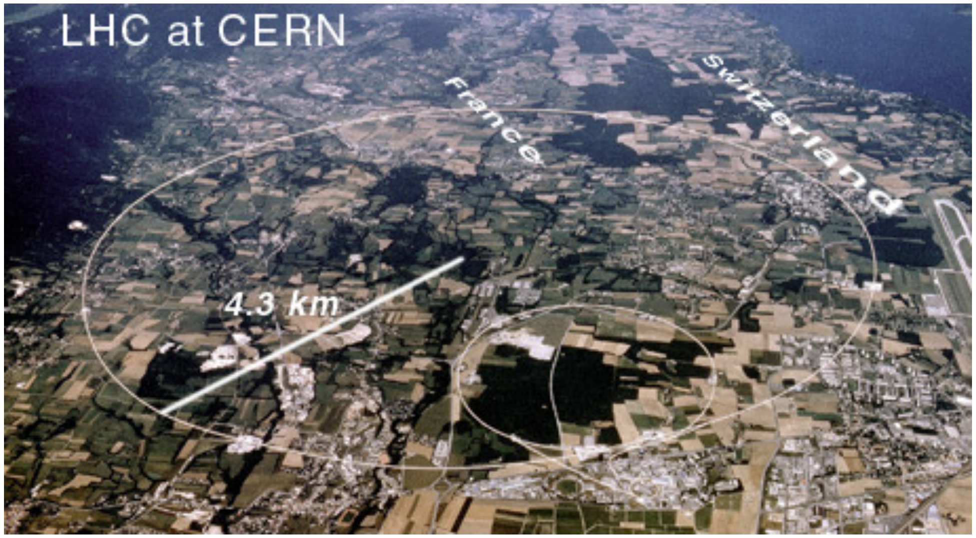
Limits of synchrotrons

$$\rho = \frac{p}{qB} \Rightarrow$$ The rings become too long

Protons with p = 20 TeV/c , B = 6.8 T would require a 87 km SSC tunnel
Protons with p = 7 TeV/c , B = 8.4 T require CERN's 27 km LHC tunnel

$$P_{\text{radiation}} = \tfrac{c}{6\pi\varepsilon_0} N \frac{q^2}{\rho^2}\gamma^4 \Downarrow$$

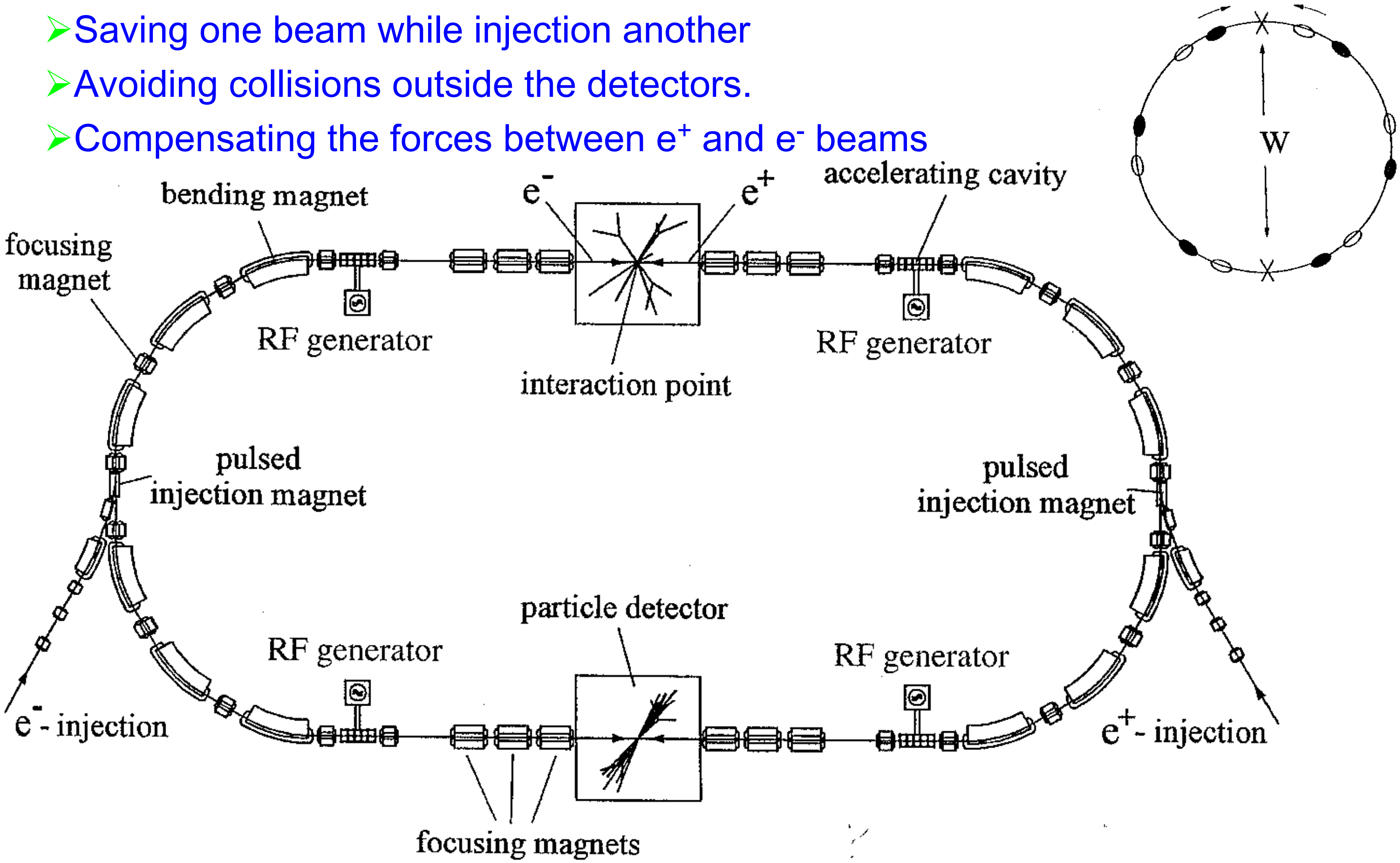
Energy needed to compensate
Radiation becomes too large



Electron beam with p = 0.1 TeV/c in CERN's 27 km LEP tunnel radiated 20 MW
Each electron lost about 4GeV per turn, requiring many RF accelerating sections.

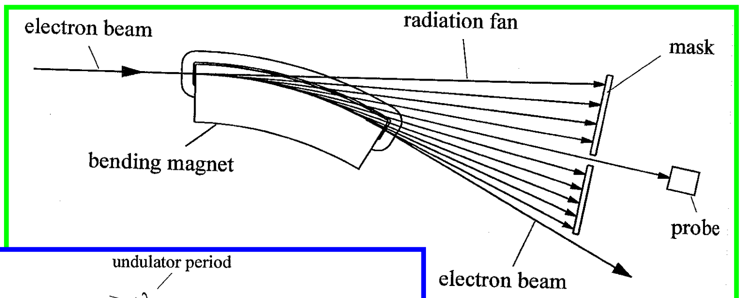
Elements of a collider in the Ring Design Tutorial

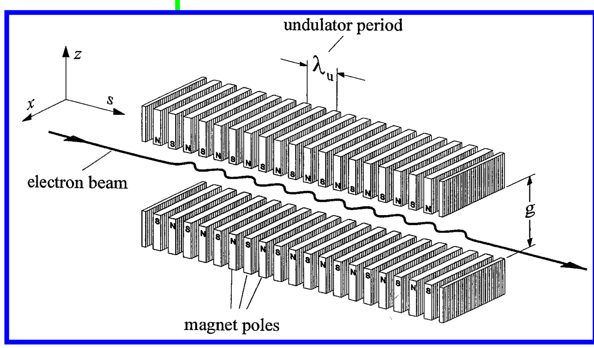
- Saving one beam while injection another
- Avoiding collisions outside the detectors.
- Compensating the forces between e^+ and e^- beams

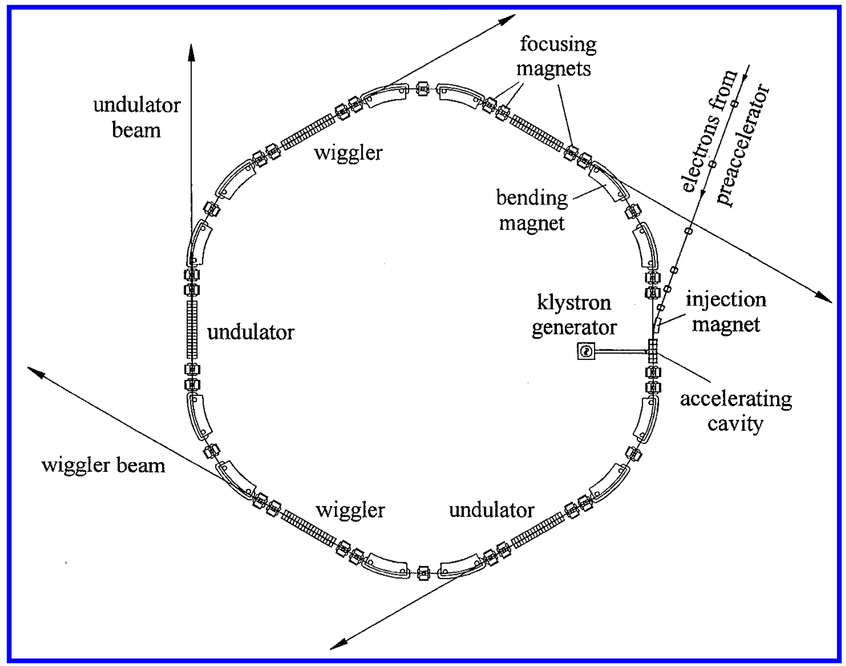


Light Sources – in Bmad – not in our Ring Design

- 1^{st} Genergation (1970s): Many HEP rings are parasitically used for X-ray production
- 2^{nd} Generation (1980s): Many dedicated X-ray sources (light sources)
- 3^{rd} Generation (1990s): Several rings with dedicated radiation devices (wigglers and undulators)
- Today (4^{th} Generation): Construction of Free Electron Lasers (FELs) driven by LINACs







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This week

Framework for tutorial work

9am: Start the day with optics background for the day's tutorial

10:30am: Coffee break

10:45am: Individual or in teams go through the Ring Design tutorial.

Monday: Fodo Cell - Dispersion compression – Twiss parameter matching

Tuesday: Constructing the ring – Low beta interaction region – Tune cell

Wednesday: RF cavities – Long Term Tracking – Dynamics aperture optimization

Thursday: Radiation – Orbit correction

Friday: 9:00 to 12:15 – Catchup time – topics upon request

- Different people will progress with different speeds. Individual extra topics can be developed with those who progress more quickly.

The Ring Design Tutorial 1 – FODO Cells

Collider Ring Design Tutorial using Bmad

July 26, 2024

Georg Hoffstaetter de Torquat
Daria Kuzovkova
David Sagan
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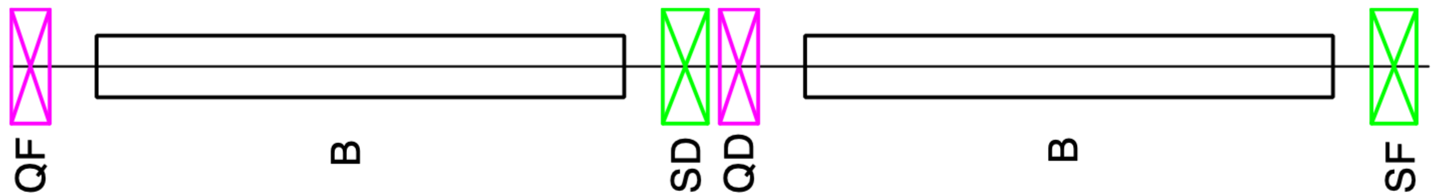
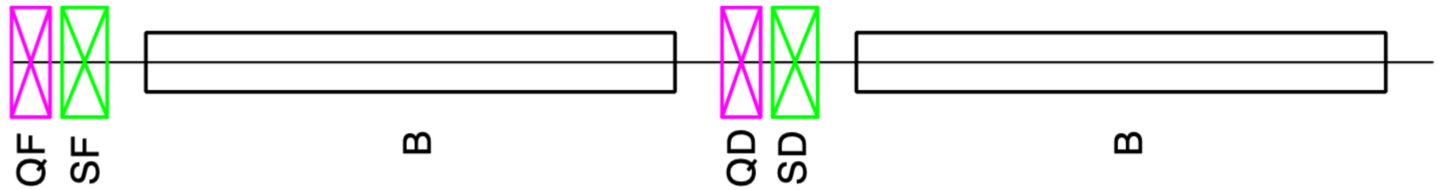
The Ring Design Tutorial 1 – FODO Cells

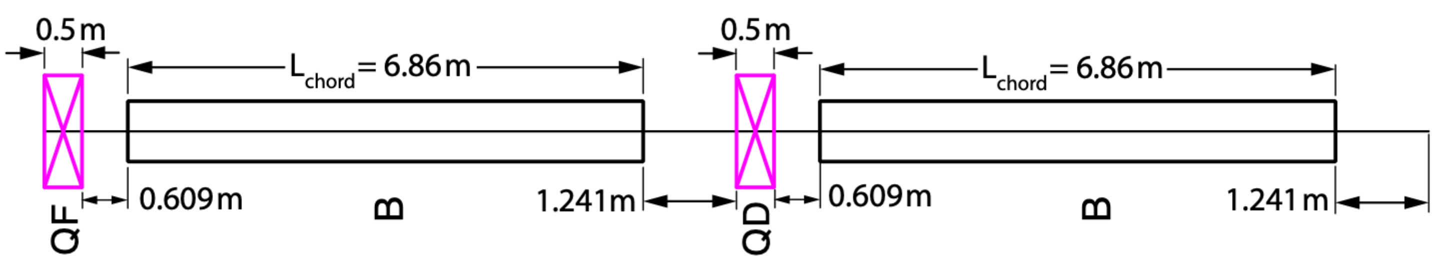
FODO Cells need quadrupoles, which makes them not mirror symmetric.



Forward FODO cell

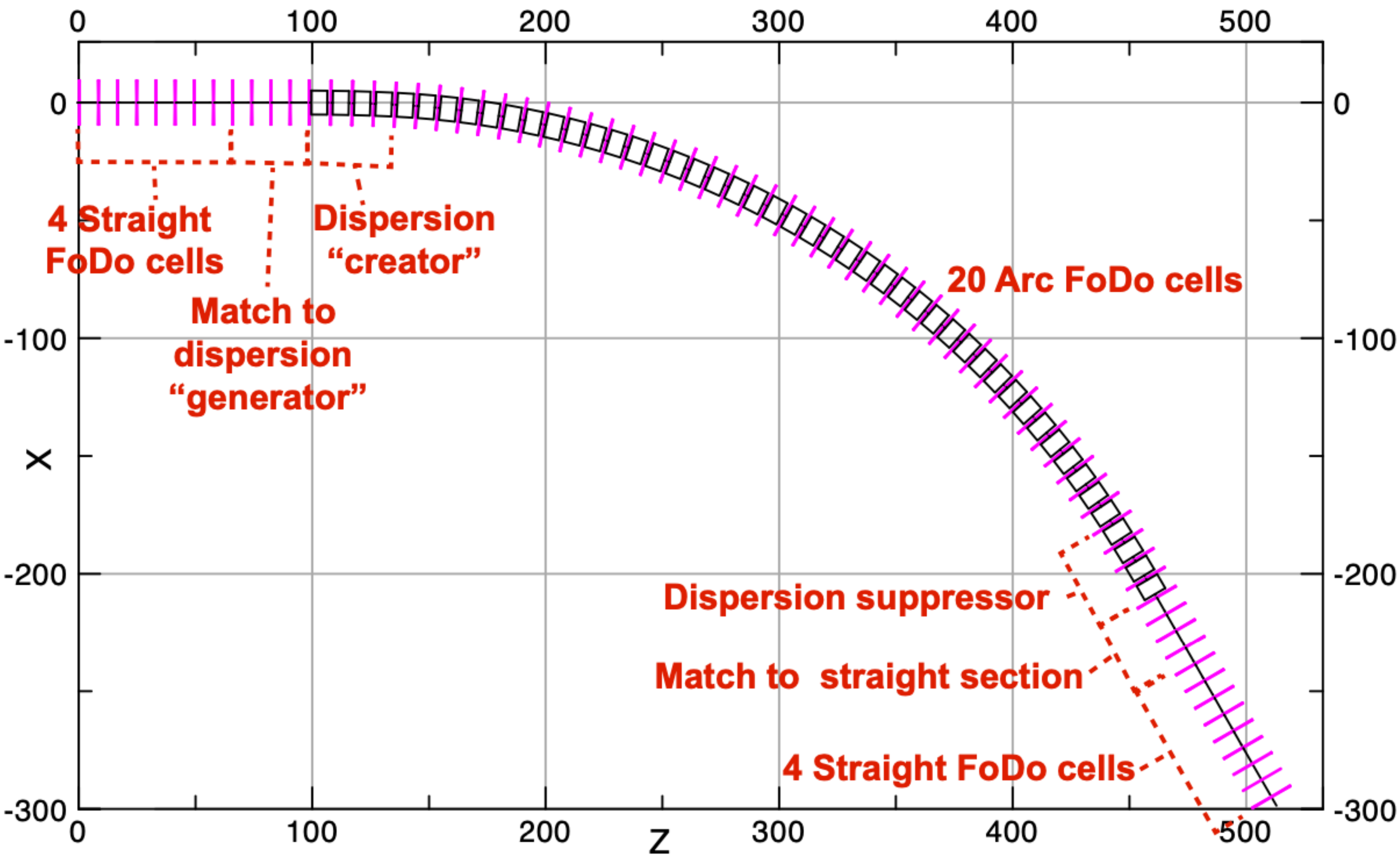


Backward FODO cell

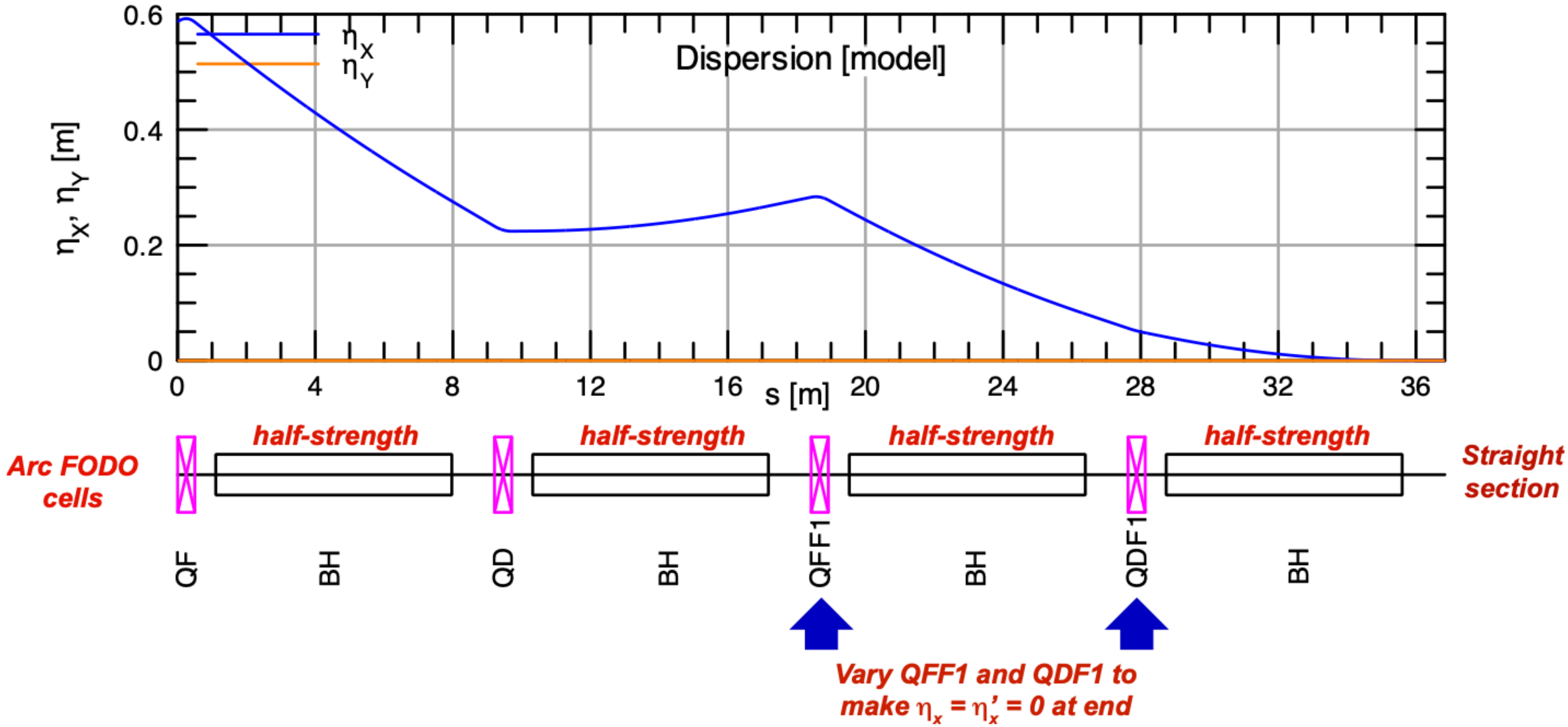


Dimensions of the Tutorial (characterizing the ESR)

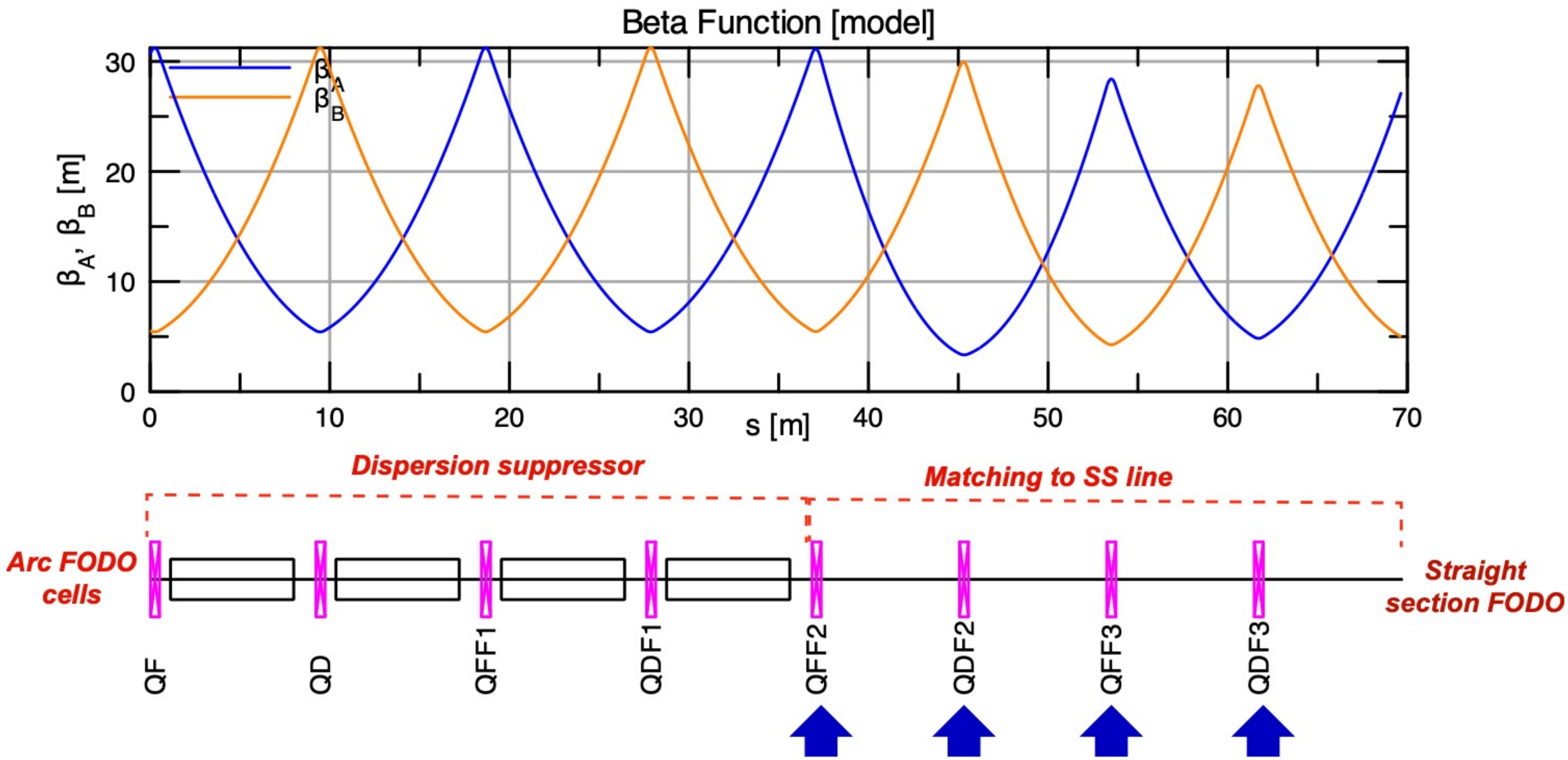
Tutorial 2 – Dispersion Suppressor



Missing Bend Dispersion Suppressors



Matching Dispersion Suppressor to FODOs

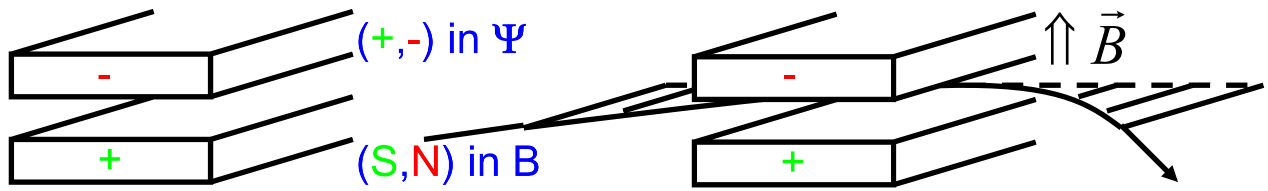


Multipoles in Accelerators: ν=1, Dipoles

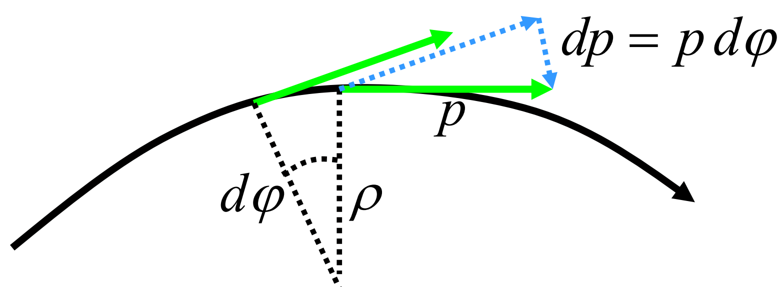
$$\psi = \Psi_1 \,\mathrm{Im}\{x - iy\} = -\Psi_1 \cdot y \quad \Rightarrow \quad \vec{B} = -\vec{\nabla}\psi = \Psi_1 \,\vec{e}_y$$

Equipotential
$y = \mathrm{const.}$

C$_1$ Symmetry



Dipole magnets are used for steering the beams direction

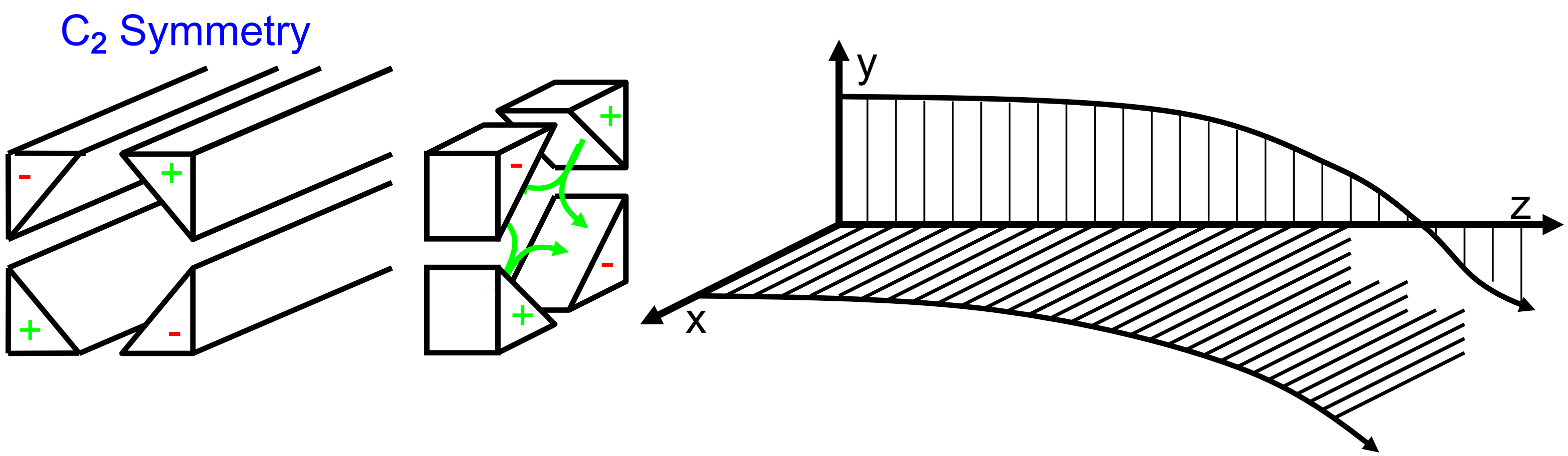


$$\frac{d\vec{p}}{dt} = q\,\vec{v} \times \vec{B} \quad \Rightarrow \quad \frac{dp}{dt} = qvB_\perp \quad \Rightarrow \quad \rho = \frac{dl}{d\varphi} = \frac{vdt}{dp/p} = \frac{p}{qB_\perp}$$

Bending radius: $\rho = \dfrac{p}{qB}$

Multipoles in Accelerators: ν=2, Quadrupoles

$$\psi = \Psi_2 \,\mathrm{Im}\{(x - iy)^2\} = -\Psi_2 \cdot 2xy \quad \Rightarrow \quad \vec{B} = -\vec{\nabla}\psi = \Psi_2\, 2\begin{pmatrix} y \\ x \end{pmatrix}$$



In a quadrupole particles are focused in one plane and defocused in the other plane. Other modes of strong focusing are not possible.

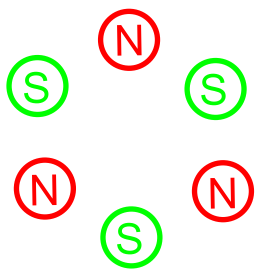
Real Quadrupoles

The coils show that this is an upright quadrupole not a rotated or skew quadrupole.

Multipoles in Accelerators: ν=3, Sextupoles

$$\psi = \Psi_3 \,\mathrm{Im}\{(x-iy)^3\} = \Psi_3 \cdot (y^3 - 3x^2 y) \quad \Rightarrow \quad \vec{B} = -\vec{\nabla}\psi = \Psi_3 \, 3 \begin{pmatrix} 2xy \\ x^2 - y^2 \end{pmatrix}$$

C_3 Symmetry



i) Sextupole fields hardly influence the particles close to the center, where one can linearize in x and y.

ii) In linear approximation a by Δx shifted sextupole has a quadrupole field.

iii) When Δx depends on the energy, one can build an energy dependent quadrupole.

$$\vec{B} = -\vec{\nabla}\psi = \Psi_3 \, 3 \begin{pmatrix} 2xy \\ x^2 - y^2 \end{pmatrix}$$

$$x \mapsto \Delta x + x$$

$$\vec{B} \approx \Psi_3 \, 3 \begin{pmatrix} 2xy \\ x^2 - y^2 \end{pmatrix} + 6\Psi_3 \Delta x \begin{pmatrix} y \\ x \end{pmatrix} + O(\Delta x^2)$$

Real Sextupoles

Higher-order multipoles

$$\psi = \Psi_n \,\mathrm{Im}\{(x-iy)^n\} = \Psi_n \cdot (\ldots - i\,n\,x^{n-1}y) \quad \Rightarrow \quad \vec{B}(y=0) = \Psi_n\, n \begin{pmatrix} 0 \\ x^{n-1} \end{pmatrix}$$

Multipole strength: $k_n = \frac{q}{p}\partial_x^n B_y\big|_{x,y=0} = \frac{q}{p}\Psi_{n+1}\,(n+1)!$ units: $\frac{1}{\mathrm{m}^{n+1}}$

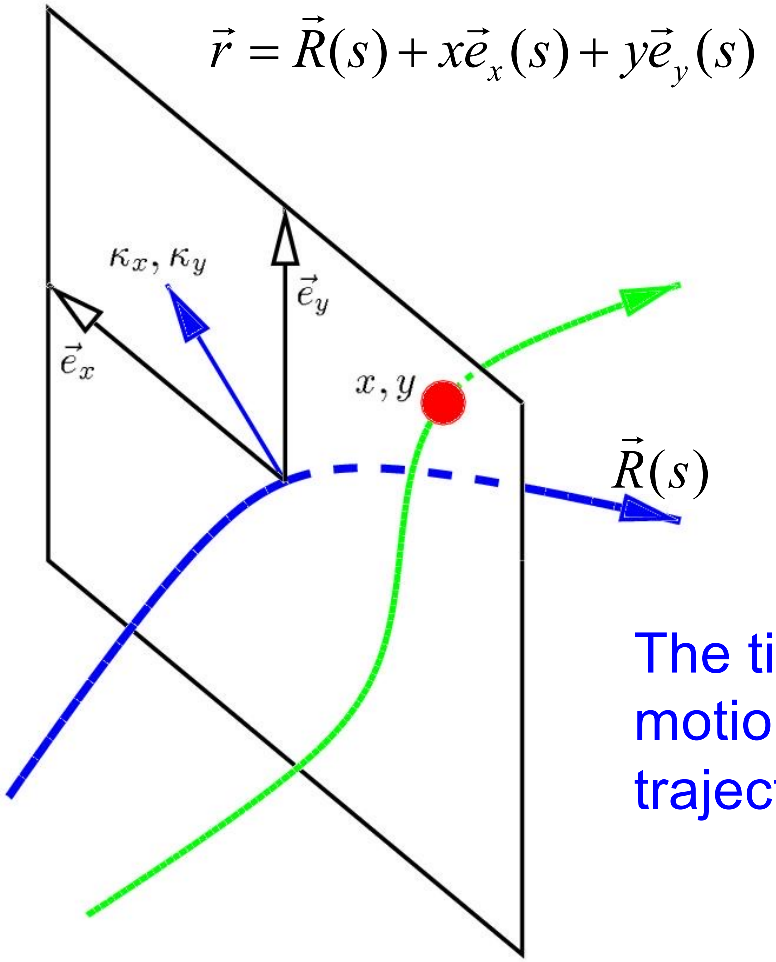
p/q is also called Bρ and used to describe the energy of multiply charge ions

Names: dipole, quadrupole, sextupole, octupole, decapole, duodecapole, …

Higher order multipoles come from

- Field errors in magnets
- Magnetized materials
- From multipole magnets that compensate such erroneous fields
- To compensate nonlinear effects of other magnets
- To stabilize the motion of many particle systems
- To stabilize the nonlinear motion of individual particles

The comoving coordinate system



$$\left|d\vec{R}\right| = ds$$

$$\boxed{\vec{e}_s \equiv \tfrac{d}{ds}\vec{R}(s)}$$

The time dependence of a particle's motion is often not as interesting as the trajectory along the accelerator length "s".

6D phase space motion

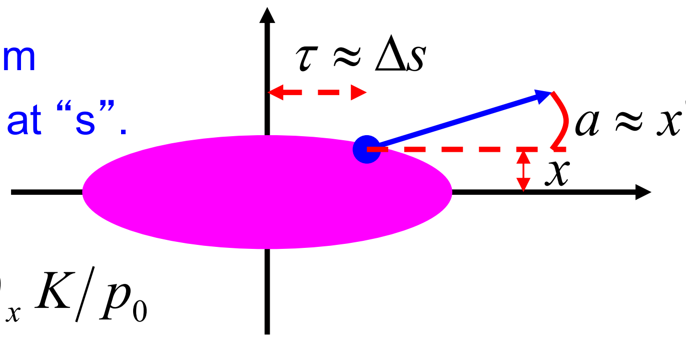
Using a reference momentum p_0 and a reference time t_0:

$$\vec{z} = (x, a, y, b, \tau, \delta)$$

$$a = \frac{p_x}{p_0}, \quad b = \frac{p_y}{p_0}, \quad \delta = \frac{E - E_0}{E_0}, \quad \tau = (t_0 - t)\frac{c^2}{v_0} = (t_0 - t)\frac{E_0}{p_0}$$

Usually p_0 is the design momentum of the beam

And t_0 is the time at which the bunch center is at “s”.



$$\left.\begin{aligned} x' &= \partial_{p_x} K \\ p_x' &= -\partial_x K \end{aligned}\right\} \Rightarrow \left\{\begin{aligned} x' &= \partial_a K / p_0 , \quad a' = -\partial_x K / p_0 \\ y' &= \partial_b K / p_0 , \quad b' = -\partial_y K / p_0 \end{aligned}\right.$$

$$-t' = \partial_E K \quad \Rightarrow \quad \tau' = \tfrac{c^2}{v_0} \partial_\delta K / E_0 = \partial_\delta K / p_0$$

$$E' = -\partial_{-t} K \quad \Rightarrow \quad \delta' = -\tfrac{1}{E_0} \partial_\tau K \tfrac{c^2}{v_0} = -\partial_\tau K / p_0$$

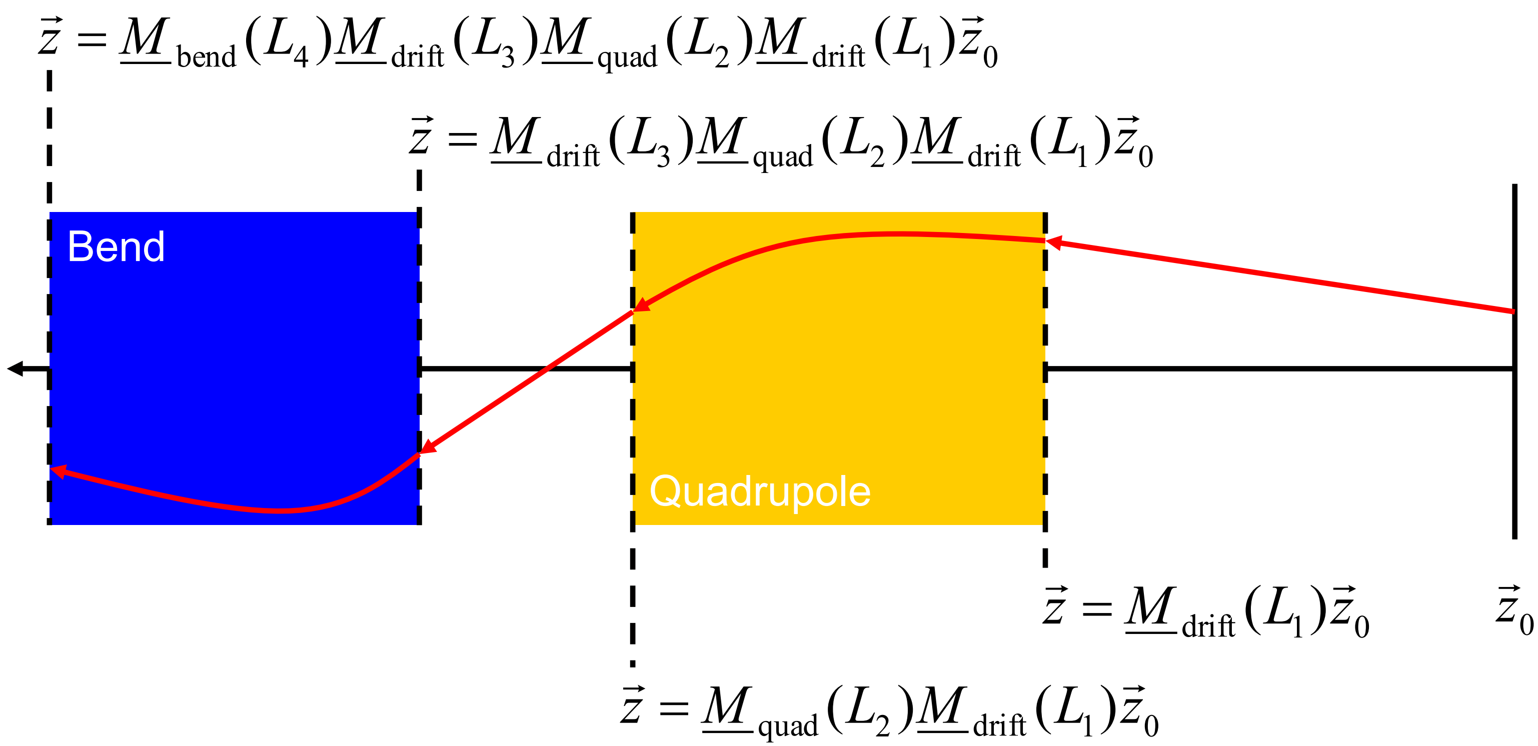
New Hamiltonian:

$$\tilde{H} = K / p_0$$

The matrix solution of linear equations of motions

Linear equation of motion: $\vec{z}' = \underline{F}(s)\vec{z}$ → $\vec{z}(s) = \underline{M}(s)\,\vec{z}_0$

Matrix solution of the starting condition $\vec{z}(0) = \vec{z}_0$



Simplest example: motion through an empty drift

$$\begin{pmatrix} x' \\ a' \\ y' \\ b' \\ \tau' \\ \delta' \end{pmatrix} = \begin{pmatrix} a \\ 0 \\ b \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\ddot{x} = 0 \quad \Rightarrow \quad x'' = 0 \quad \Rightarrow \quad a = x', a' = 0$$

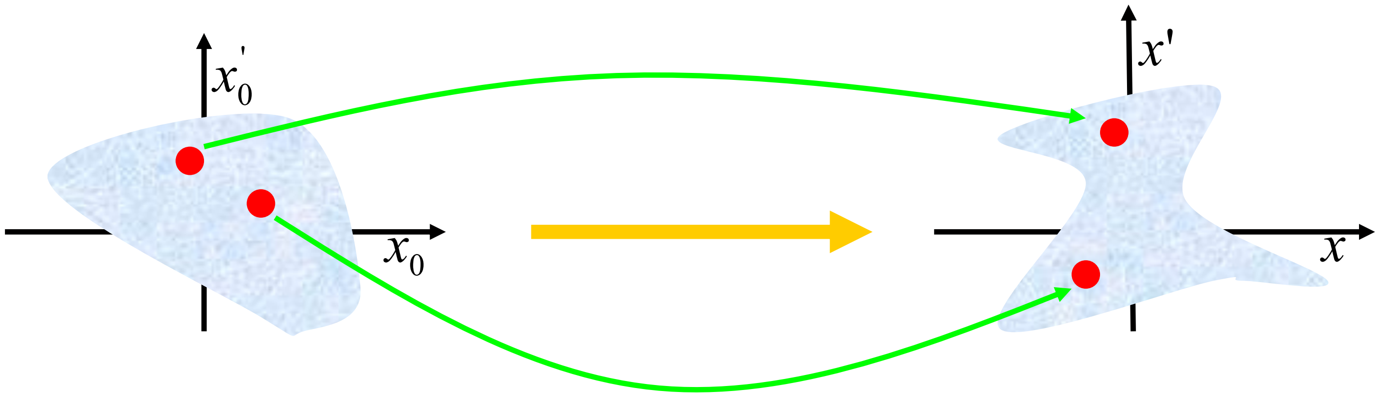
Linear solution:

$$x(s) = x_0 + x_0' s$$

$$\begin{pmatrix} x \\ a \\ y \\ b \\ \tau \\ \delta \end{pmatrix} = \begin{pmatrix} x_0 + s a_0 \\ a \\ y_0 + s b_0 \\ b_0 \\ \tau_0 \\ \delta_0 \end{pmatrix} = \begin{pmatrix} \begin{matrix} 1 & s \\ 0 & 1 \end{matrix} & \underline{0} & \underline{0} \\ \underline{0} & \begin{matrix} 1 & s \\ 0 & 1 \end{matrix} & \underline{0} \\ \underline{0} & \underline{0} & \begin{matrix} 1 & 0 \\ 0 & 1 \end{matrix} \end{pmatrix} \vec{z}_0$$

Liouville's Theorem

- A phase space volume does not change when it is transported by Hamiltonian motion. $\vec{z}(s) = \underline{M}(s)\cdot\vec{z}_0$ with $\det[\underline{M}(s)] = +1$



Volume = $V = \iint_V d^n\vec{z} = \iint_{V_0} \left|\frac{\partial\vec{z}}{\partial\vec{z}_0}\right| d^n\vec{z}_0 = \iint_{V_0} |\underline{M}| d^n\vec{z}_0 = \iint_{V_0} d^n\vec{z}_0 = V_0$

Hamiltonian Motion → $V = V_0$

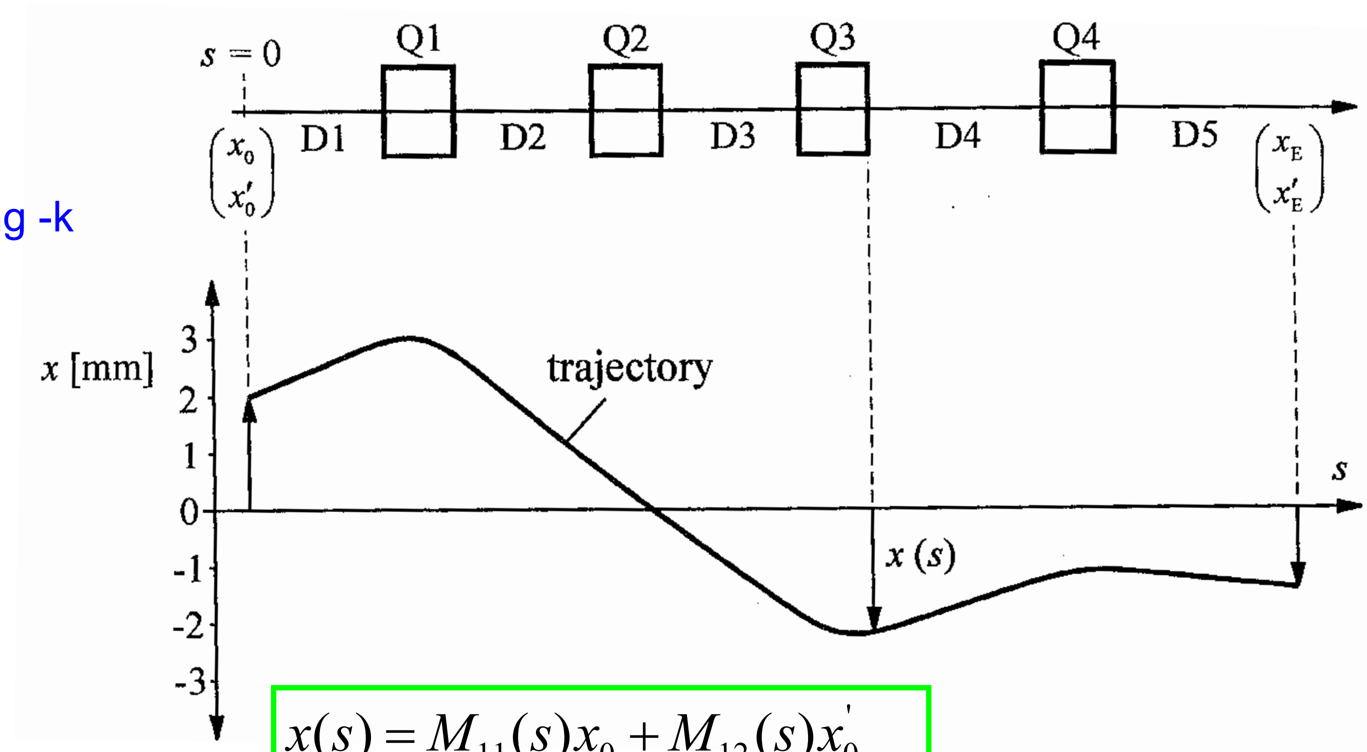
But Hamiltonian requires symplecticity, which is much more than just $\det[\underline{M}(s)] = +1$

Betatron formalism for linear motion

$$x'' = -x\,K$$
$$y'' = \quad y\,k$$

In y: quadrupole defocusing -k

In x: $K = k + \frac{1}{\rho^2}$



$$x(s) = M_{11}(s)x_0 + M_{12}(s)x_0'$$
$$x(s) = \sqrt{2J\beta(s)}\,\sin(\psi(s) + \phi_0)$$

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Twiss parameters

$$x'' = -k\,x$$

$$x(s) = \sqrt{2J\beta(s)}\,\sin(\psi(s)+\phi_0)$$

$$x'(s) = \sqrt{\tfrac{2J}{\beta}}[\beta\psi'\cos(\psi(s)+\phi_0) - \alpha\sin(\psi(s)+\phi_0)] \quad \text{with} \quad \alpha = -\tfrac{1}{2}\beta'$$

$$x''(s) = \sqrt{\tfrac{2J}{\beta}}[(\beta\psi'' - 2\alpha\psi')\cos(\psi(s)+\phi_0) - (\alpha' + \tfrac{\alpha^2}{\beta} + \beta\psi'^2)\sin(\psi(s)+\phi_0)]$$

$$= \sqrt{\tfrac{2J}{\beta}}[-k\beta\sin(\psi(s)+\phi_0)]$$

$$\beta\psi'' - 2\alpha\psi' = \beta\psi'' + \beta'\psi' = (\beta\psi')' = 0 \quad \Rightarrow \quad \psi' = \tfrac{I}{\beta}$$

$$\alpha' + \gamma = k\beta \quad \textit{with} \quad \gamma = \tfrac{I^2+\alpha^2}{\beta}$$

Universal choice: I=1!

$$\beta' = -2\alpha$$

$$\alpha' = k\beta - \gamma$$

$$\psi = \int_0^s \tfrac{I}{\beta(s')}\,ds'$$

$\alpha, \beta, \gamma, \psi$ are called Twiss parameters.

What are the initial conditions?

The phase ellipse

Particles with a common J and different ϕ all lie on an ellipse in phase space:

$$\begin{pmatrix} x \\ x' \end{pmatrix} = \sqrt{2J} \begin{pmatrix} \sqrt{\beta} & 0 \\ -\frac{\alpha}{\sqrt{\beta}} & \frac{I}{\sqrt{\beta}} \end{pmatrix} \begin{pmatrix} \sin(\psi(s)+\phi_0) \\ \cos(\psi(s)+\phi_0) \end{pmatrix}$$

(Linear transform of a circle)

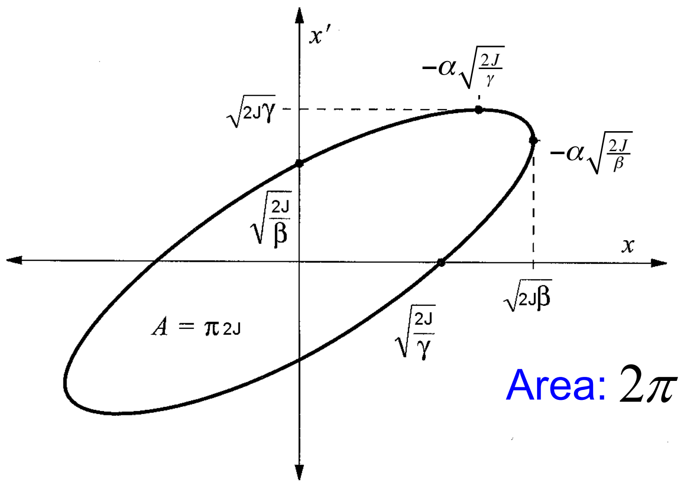
$$x_{\max} = \sqrt{2J\beta} \text{ at } x' = -\alpha\sqrt{\frac{2J}{\beta}}$$

$$(x,x') \begin{pmatrix} \frac{I}{\sqrt{\beta}} & \frac{\alpha}{\sqrt{\beta}} \\ 0 & \sqrt{\beta} \end{pmatrix} \begin{pmatrix} \frac{I}{\sqrt{\beta}} & 0 \\ \frac{\alpha}{\sqrt{\beta}} & \sqrt{\beta} \end{pmatrix} \begin{pmatrix} x \\ x' \end{pmatrix} = (x,x') \begin{pmatrix} \gamma & \alpha \\ \alpha & \beta \end{pmatrix} \begin{pmatrix} x \\ x' \end{pmatrix} = 2J$$

(Quadratic form)

$$\beta\gamma - \alpha^2 = I^2$$

Area: $2\pi J / I$



I=1 is therefore a useful choice!

What β is for x, γ is for x'

$$x'_{\max} = \sqrt{2J\gamma} \text{ at } x = -\alpha\sqrt{\frac{2J}{\gamma}}$$

Area: $2\pi J$ →

$$\int_0^{2\pi}\int_0^{J} dJ d\phi = 2\pi J = \iint dx dx'$$

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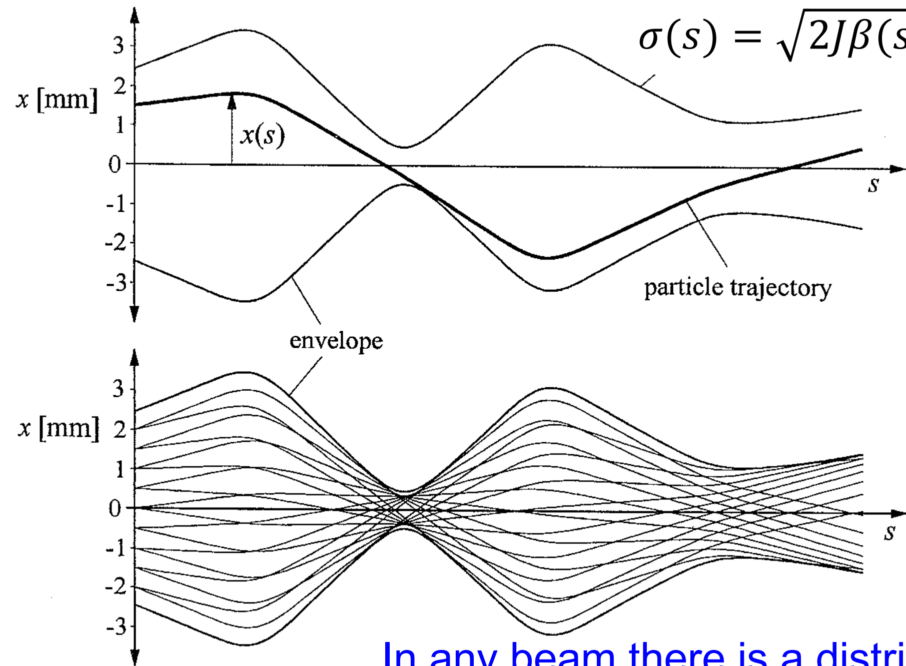
The beam envelope



$$x(s) = \sqrt{2J\beta(s)}\,\sin(\psi(s) + \phi_0)$$

$$\sigma' = -\alpha\sqrt{\frac{2J}{\beta}}$$

$$\sigma'' = -(k\beta - \gamma)\sqrt{\frac{2J}{\beta}} - \alpha^2\sqrt{\frac{2J}{\beta^3}}$$

$$\sigma'' = -(k\beta^2 - 1)\sqrt{\frac{2J}{\beta^3}}$$

In any beam there is a distribution of initial parameters. If the particles with the largest J are distributed in ϕ over all angles, then the envelope of the beam is described by $\sigma = \sqrt{2J_{max}\beta(s)}$.

The initial conditions of β and α are chosen so that this is approximately the case.

The envelope equation: $\sigma'' = -k\sigma + \frac{(2J)^2}{\sigma^3}$

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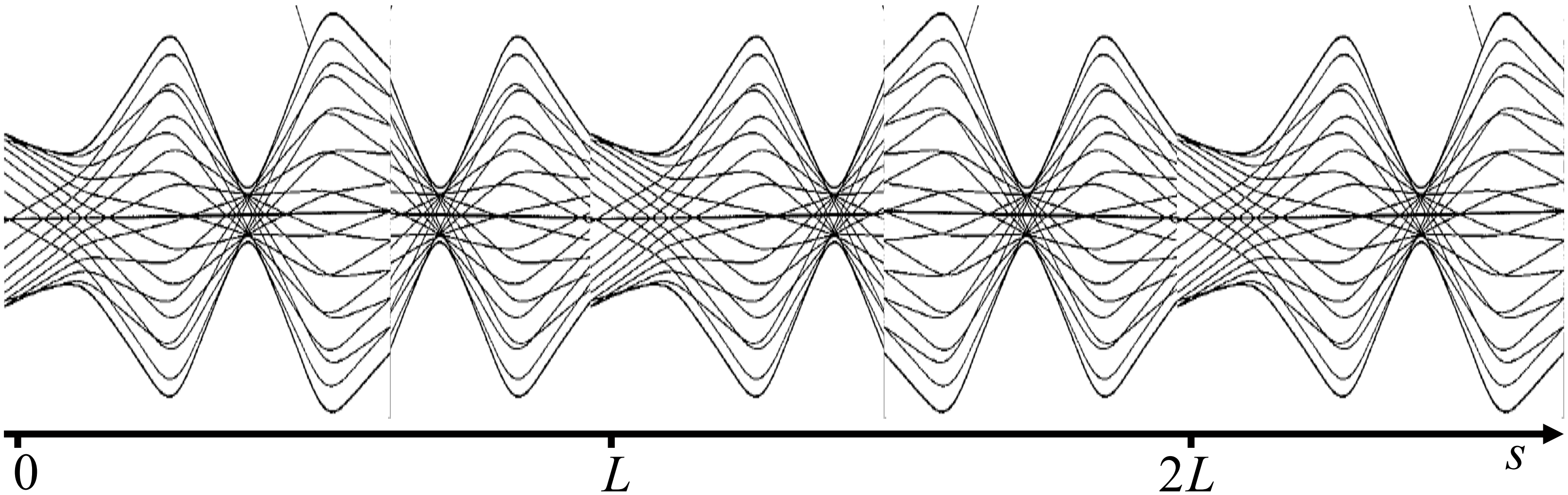
Periodic solutions in a periodic accelerator



$$\vec{z}(s) = \underline{M}(s,0)\vec{z}(0)$$

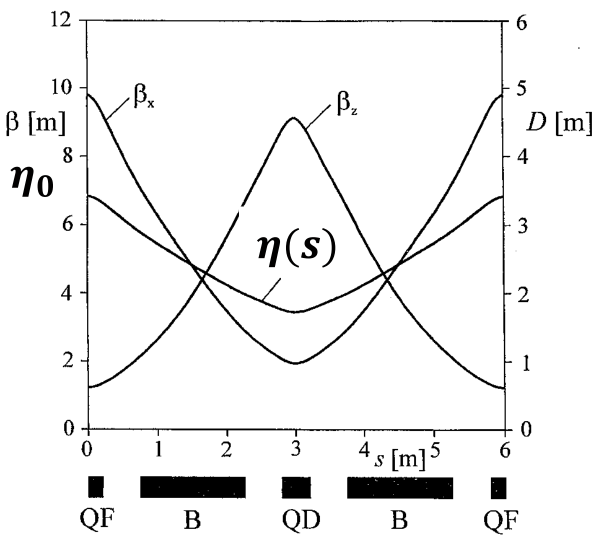
$$\vec{z}(L) = \underline{M}(L,0)\vec{z}(0)$$

$$\vec{z}(s+L) = \underline{M}_0(s)\vec{z}(s) \quad , \quad \underline{M}_0 = \underline{M}(s+L,s)$$

$$\vec{z}(s+nL) = \underline{M}_0^n(s)\vec{z}(s)$$

Fodo Cells and periodic dispersion

Alternating gradients allow focusing in both transverse plains. Therefore, focusing and defocusing quadrupoles are usually alternated and interleaved with bending magnets.



The dispersion that starts with 0 is called $\boldsymbol{D}(\boldsymbol{s})$, the dispersion that is periodic in a section is called $\boldsymbol{\eta}(\boldsymbol{s})$.

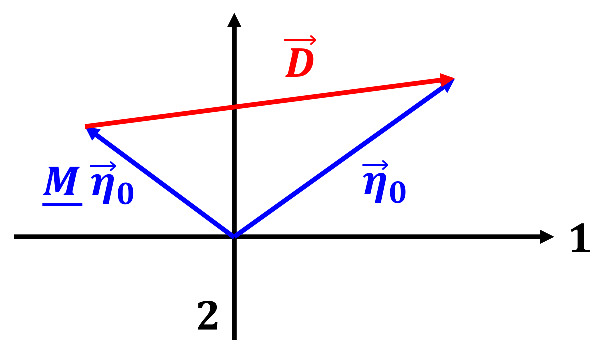
$$\underline{M}_0 = \underline{M}_{FoDo}^N$$

The periodic beta function and dispersion for each FODO is also periodic for an accelerator section that consists of many FODO cells. Often large sections of an accelerator consist of FODOs.

$$\vec{\eta}\,(s) = \begin{pmatrix} \eta(s) \\ \eta'(s) \end{pmatrix}, \vec{D}\,(s) = \begin{pmatrix} D(s) \\ D'(s) \end{pmatrix}$$

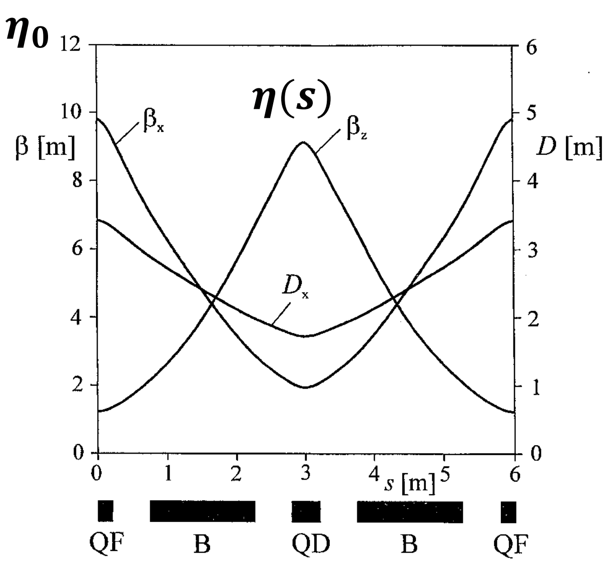
$$\vec{\eta}\,(s) = \underline{M}(s)\,\vec{\eta}_0 + \vec{D}\,(s)$$

After the FoDo: $\vec{\eta}_0 = \underline{M}(s)\,\vec{\eta}_0 + \vec{D}$ ➔ $\vec{\eta}_0 = (\underline{1} - \underline{M})^{-1}\,\vec{D}$

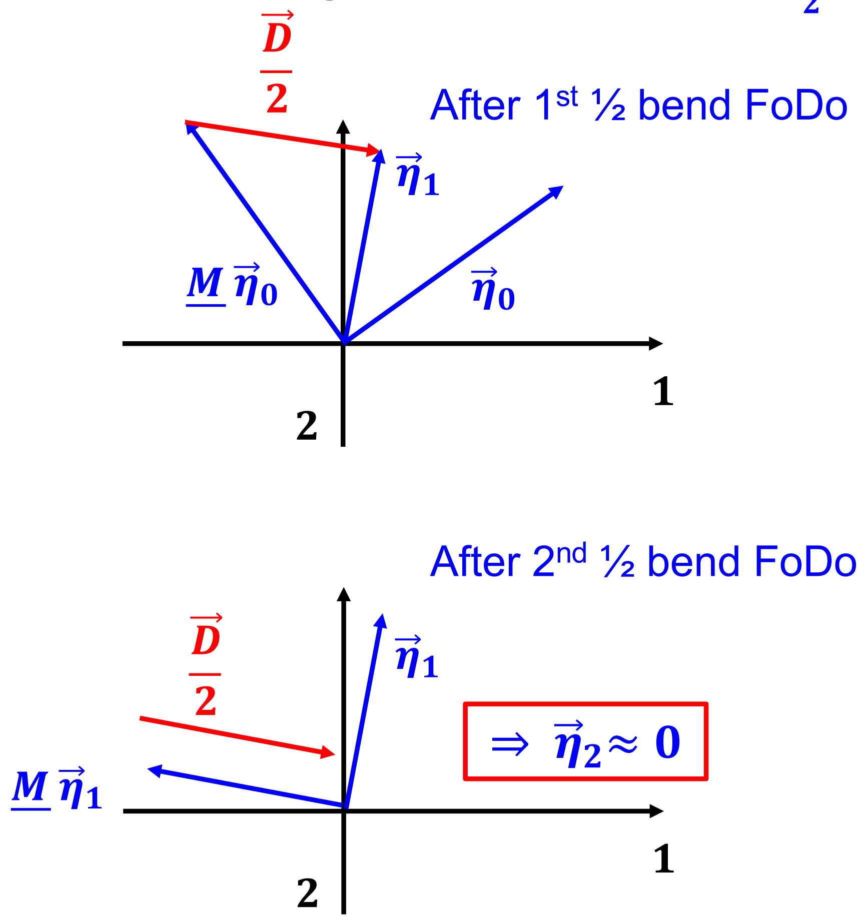


Dispersion suppression by missing bends

After an arc of periodic FoDo cells one would often like to suppress the dispersion to 0 while not changing the betas much from the periodic cells. This can be done by having one FoDo with bends reduced by a factor α followed by one reduced by (1- α).



Example: **90 degrees** FoDo cell and $\alpha = \frac{1}{2}$:



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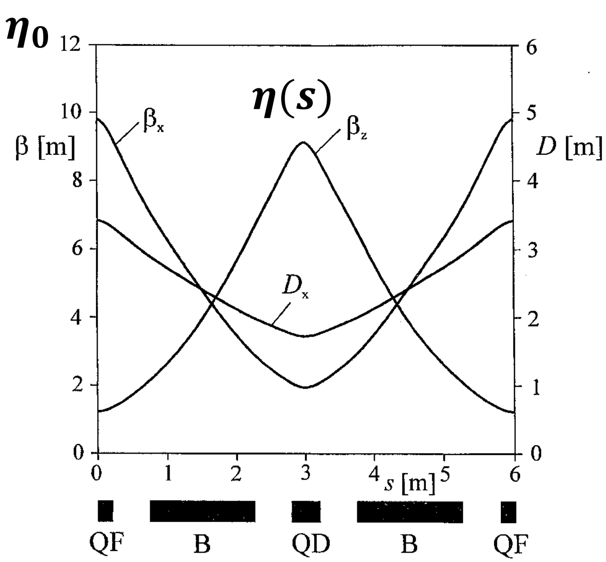
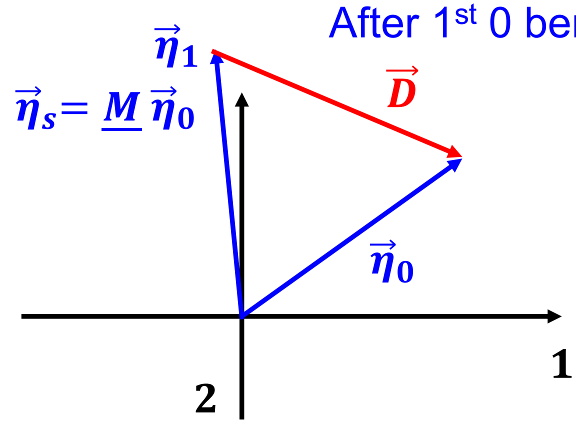
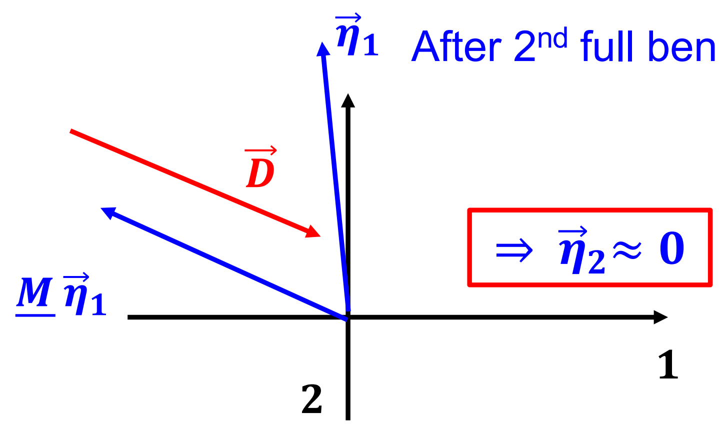


Example: **60 degrees** FoDo cell and $\alpha = 0$:

After 1st 0 bend FoDo



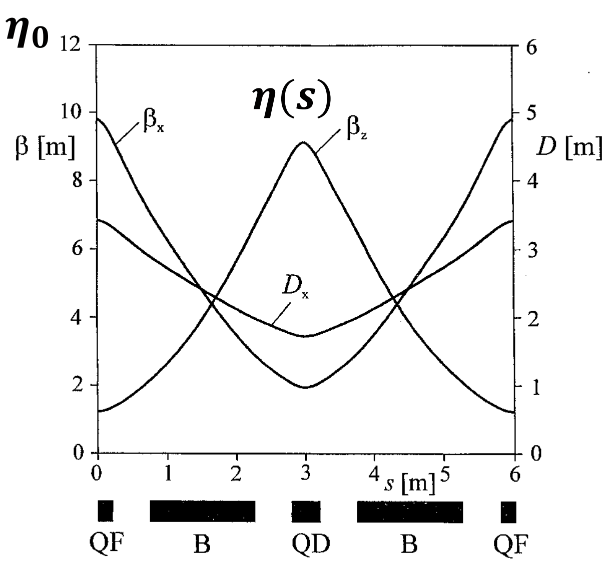
After 2nd full bend FoDo



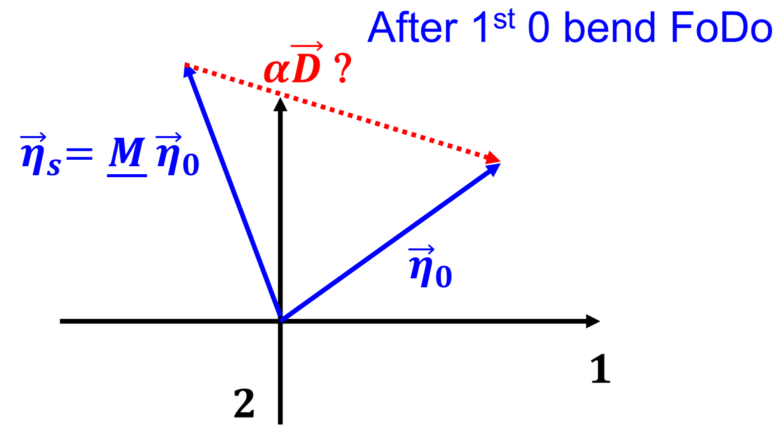
For every FoDo phase advance there is an α to make η 0.

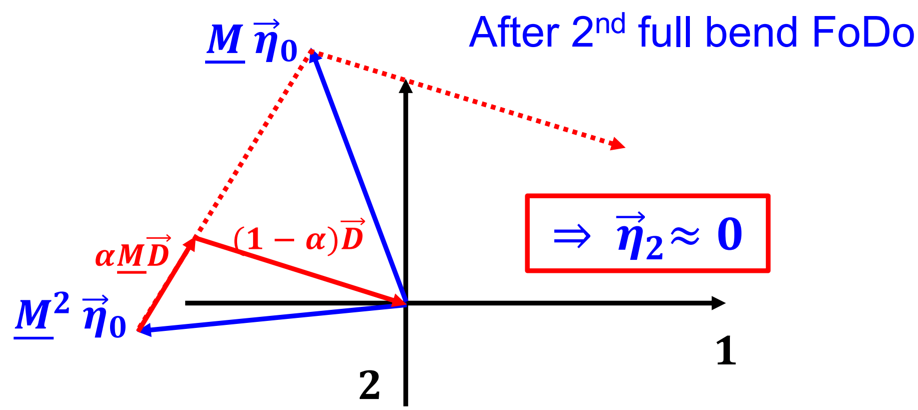
Dispersion suppression by missing bends

After an arc of periodic FoDo cells one would often like to suppress the dispersion to 0 while not changing the betas much from the periodic cells. This can be done by having one FoDo with bends reduced by a factor α followed by one reduced by (1- α).



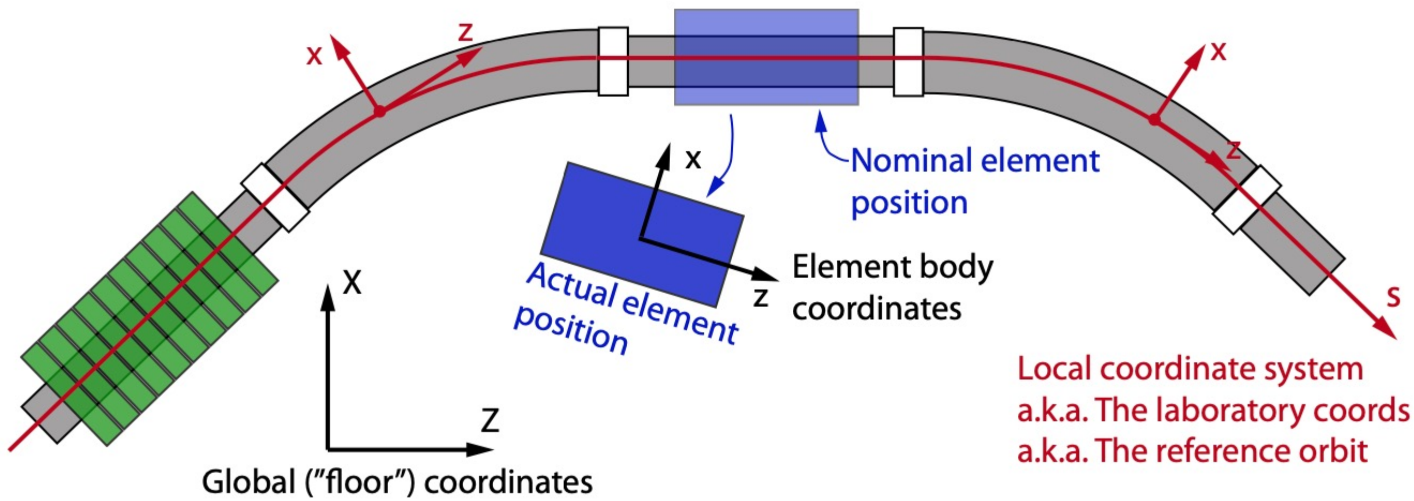
For any other FoDo phase advance: is there an $\boldsymbol{\alpha}$?





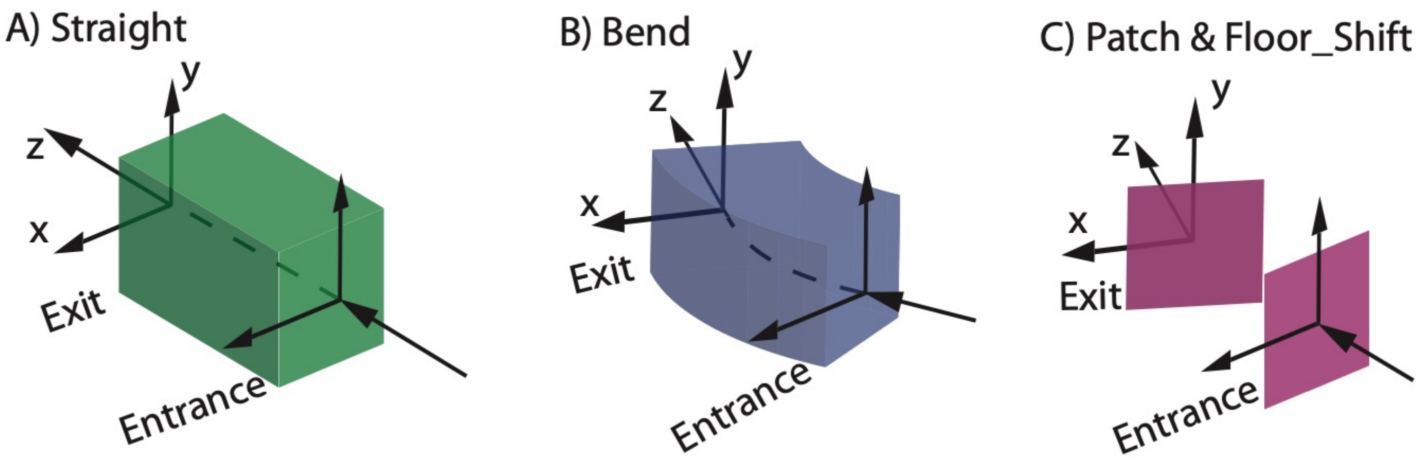
For every FoDo phase advance there is an α to make η 0.

Coordinate systems in Bmad



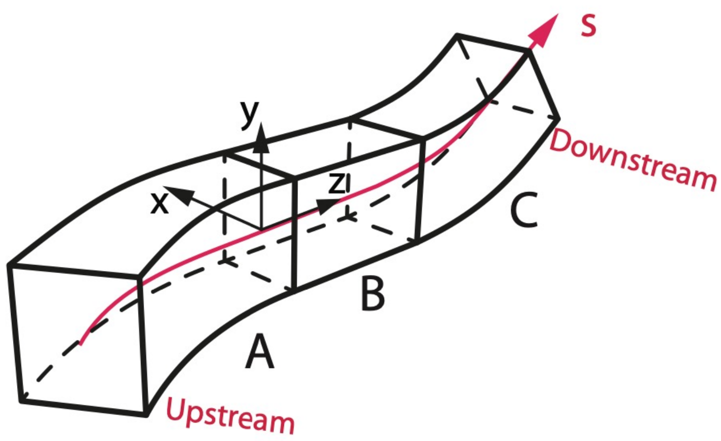
The three coordinate systems used to describe lattice element positioning: **Global**, **reference**, and **element body** coordinates.

Propagation of the coordinate systems

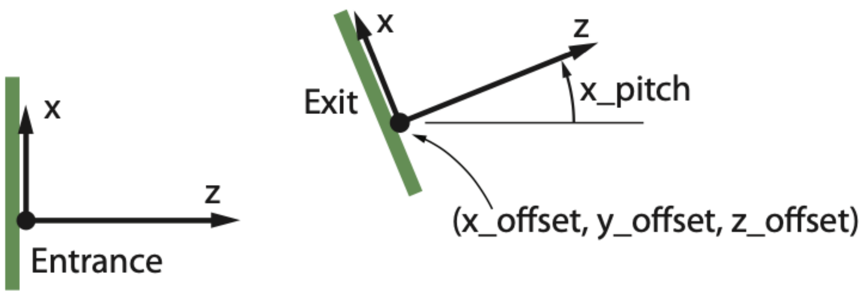


Lattice element geometry types: **Straight**, **bend**, and **patch**. All elements have an **entrance** coordinate system and an **exit** coordinate system.

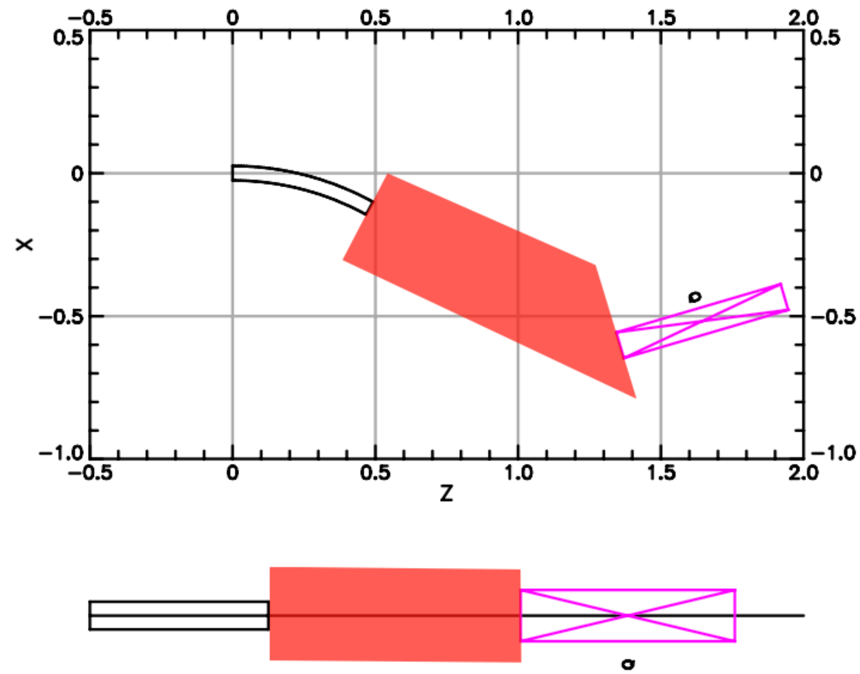
The **local** coordinate system is constructed by taking the ordered list of lattice elements and connecting the **exit** frame of one element to the **entrance** frame of the next.



Continuation after a patch element

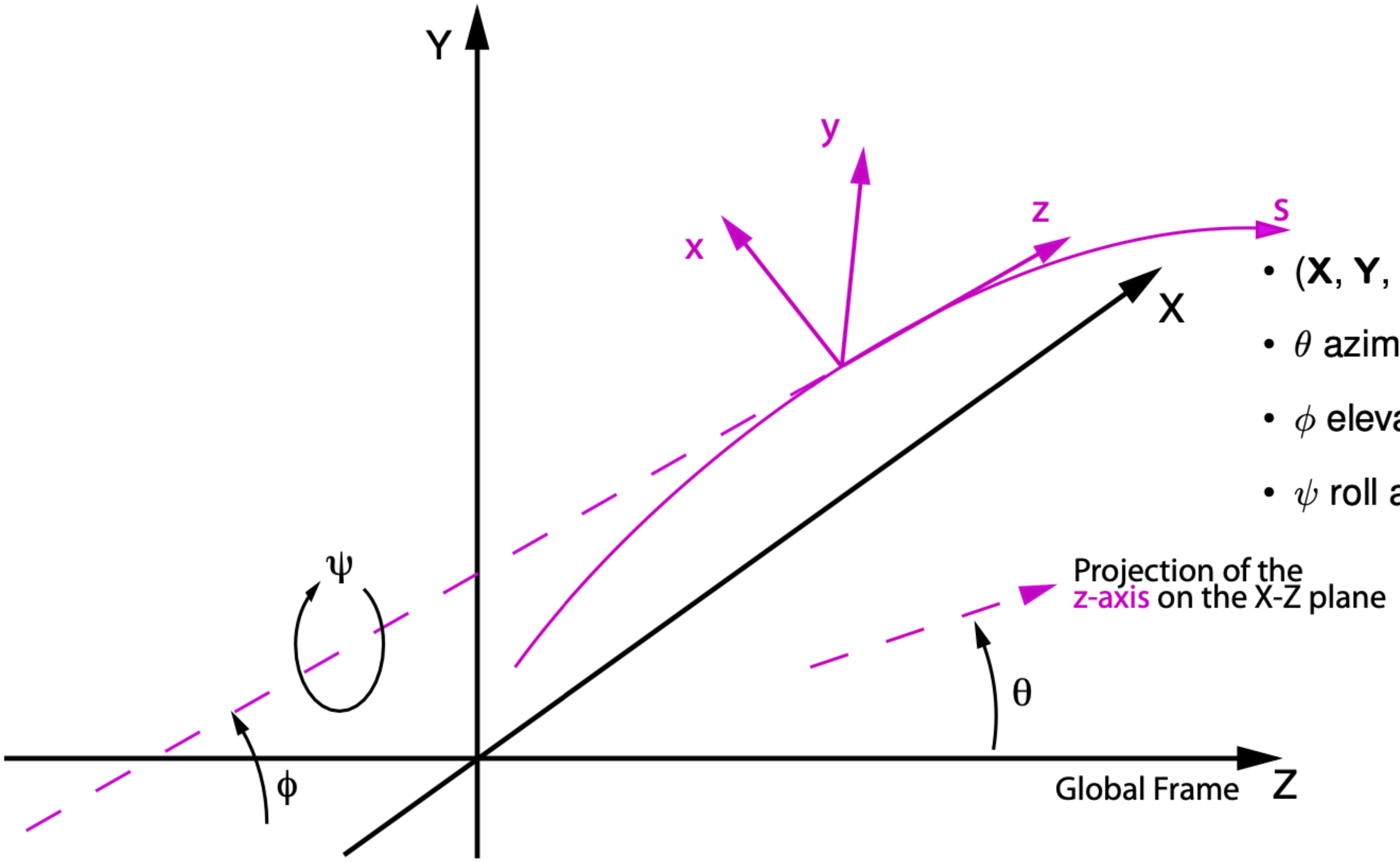


(a) The body coordinates at the exit end of a **patch** is set by the element attributes **x_offset**, **y_offset**, **z_offset**, **x_pitch**, **y_pitch**, and **tilt**.



(b) Lattice with a patch element. The patch element is the coordinates patch element in a lattice. [Note: The default is not to draw **patch** elements in a **floor_plan** plot.]

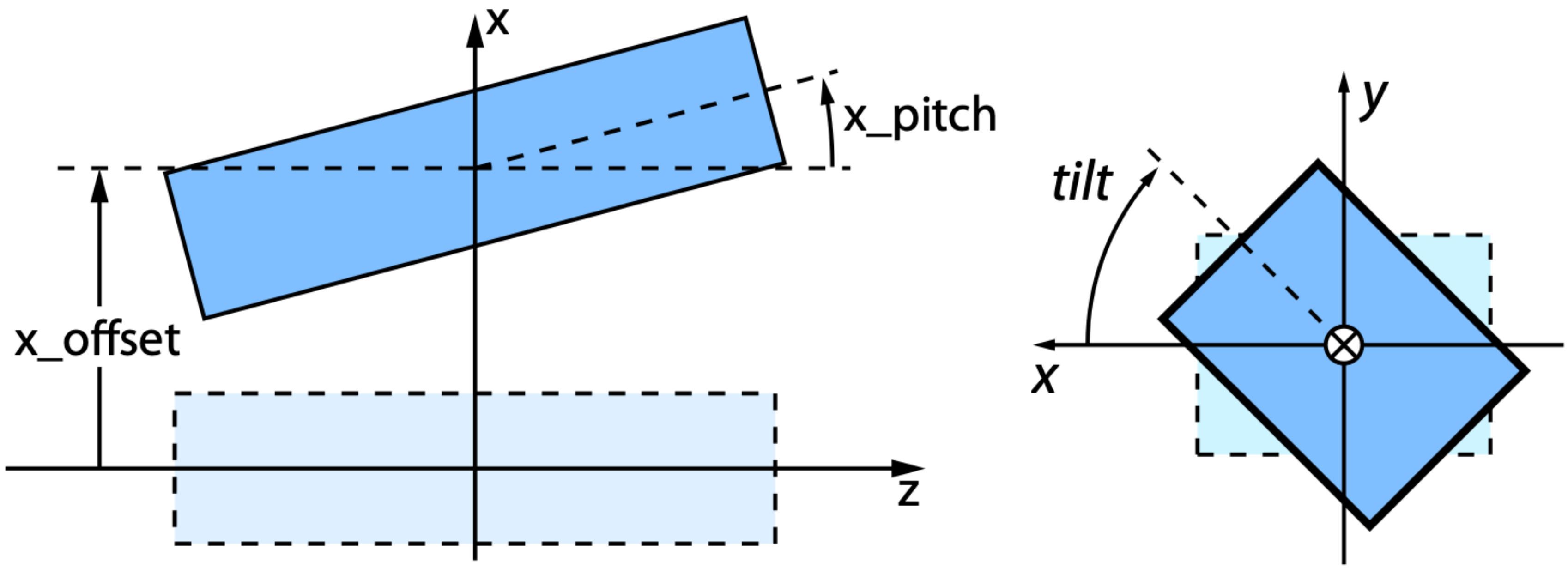
Bmad's global coordinates



- (**X**, **Y**, **Z**) global position
- θ azimuth angle in the (X, Z) plane.
- ϕ elevation angle
- ψ roll angle.

The global coordinate system.

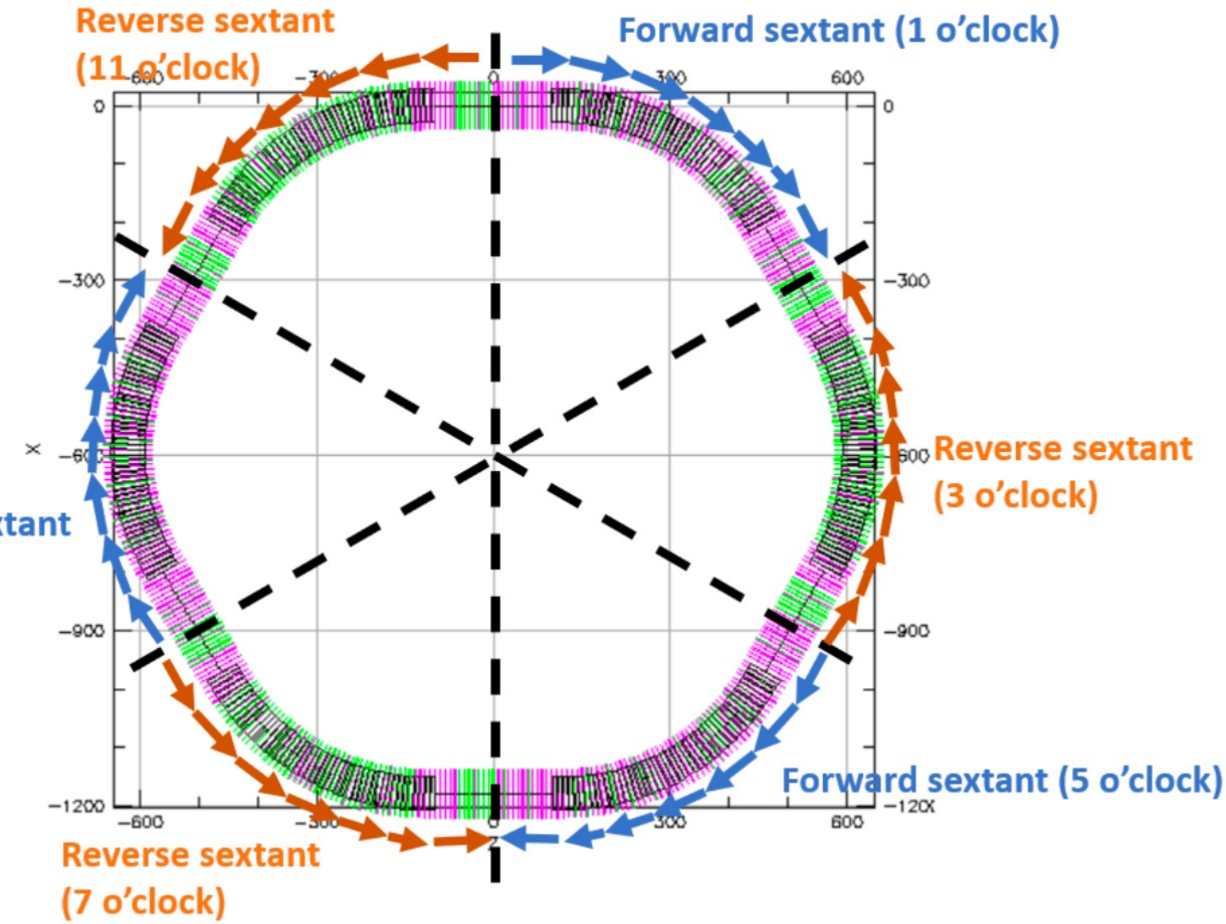
Element alignment (local to body coordinates)



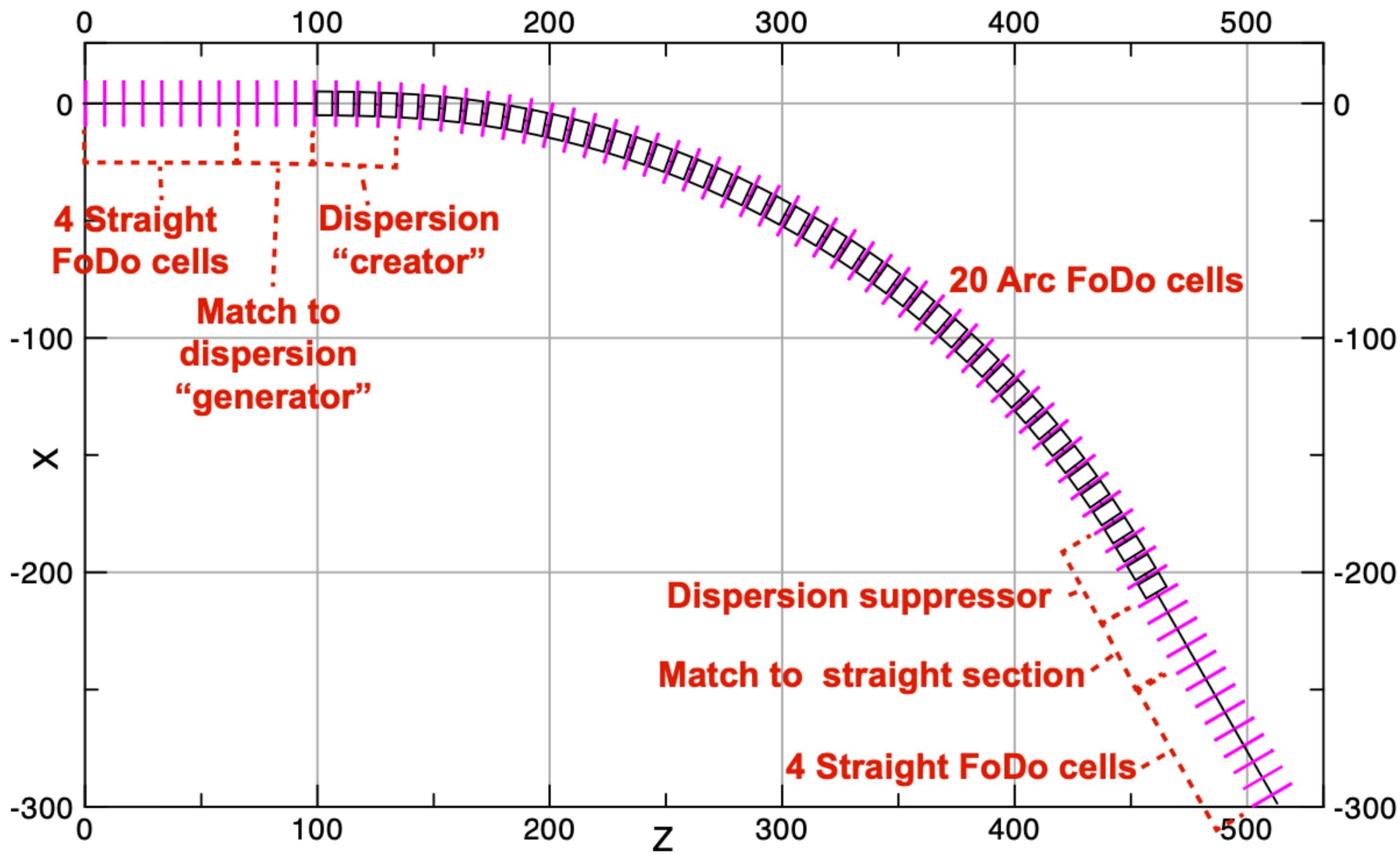
(a) Effect of x_offset and x_pitch on a straight line element

(b) Effect of a tilt on a straight line element.

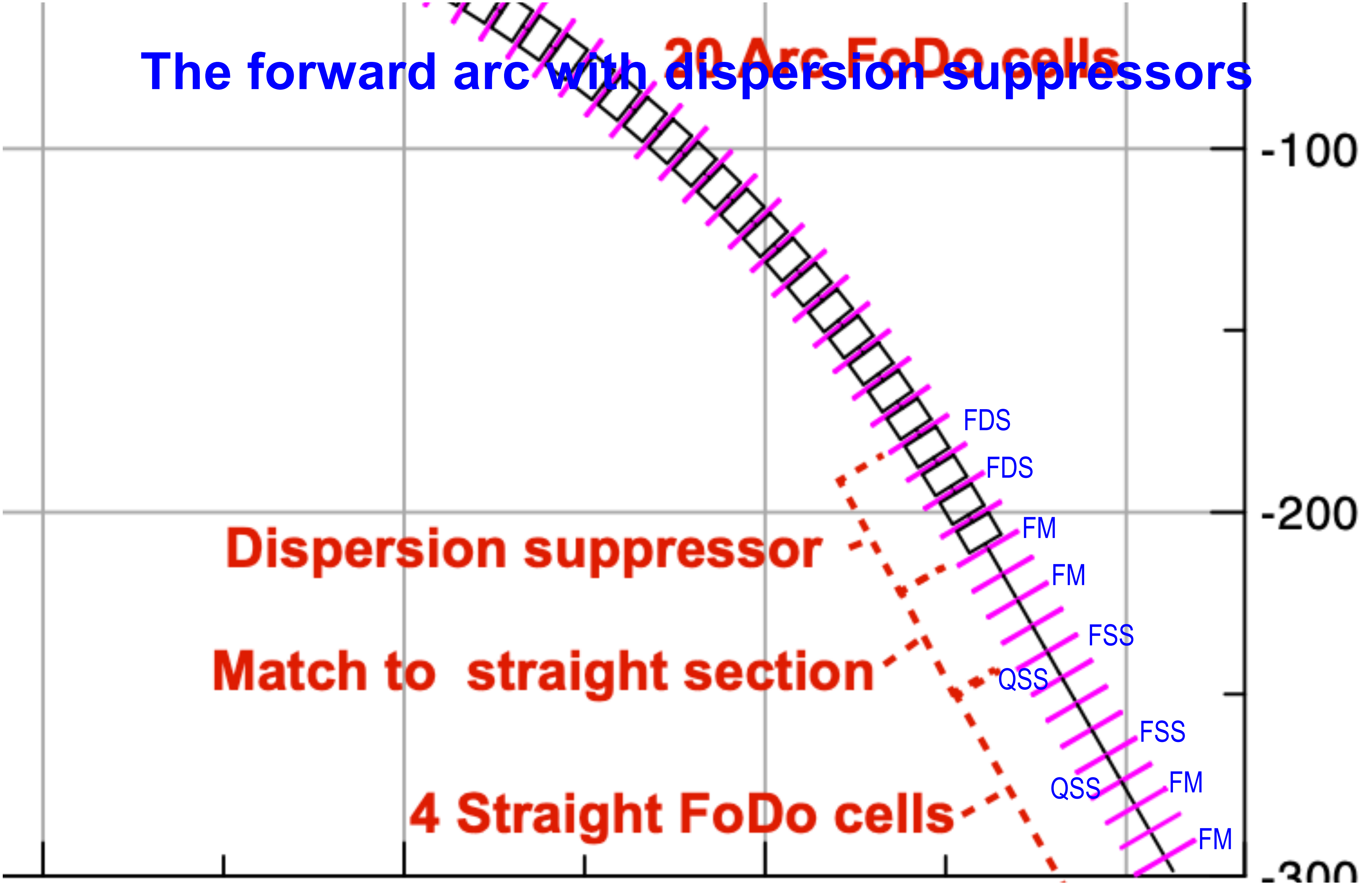
Constructing the bare ring



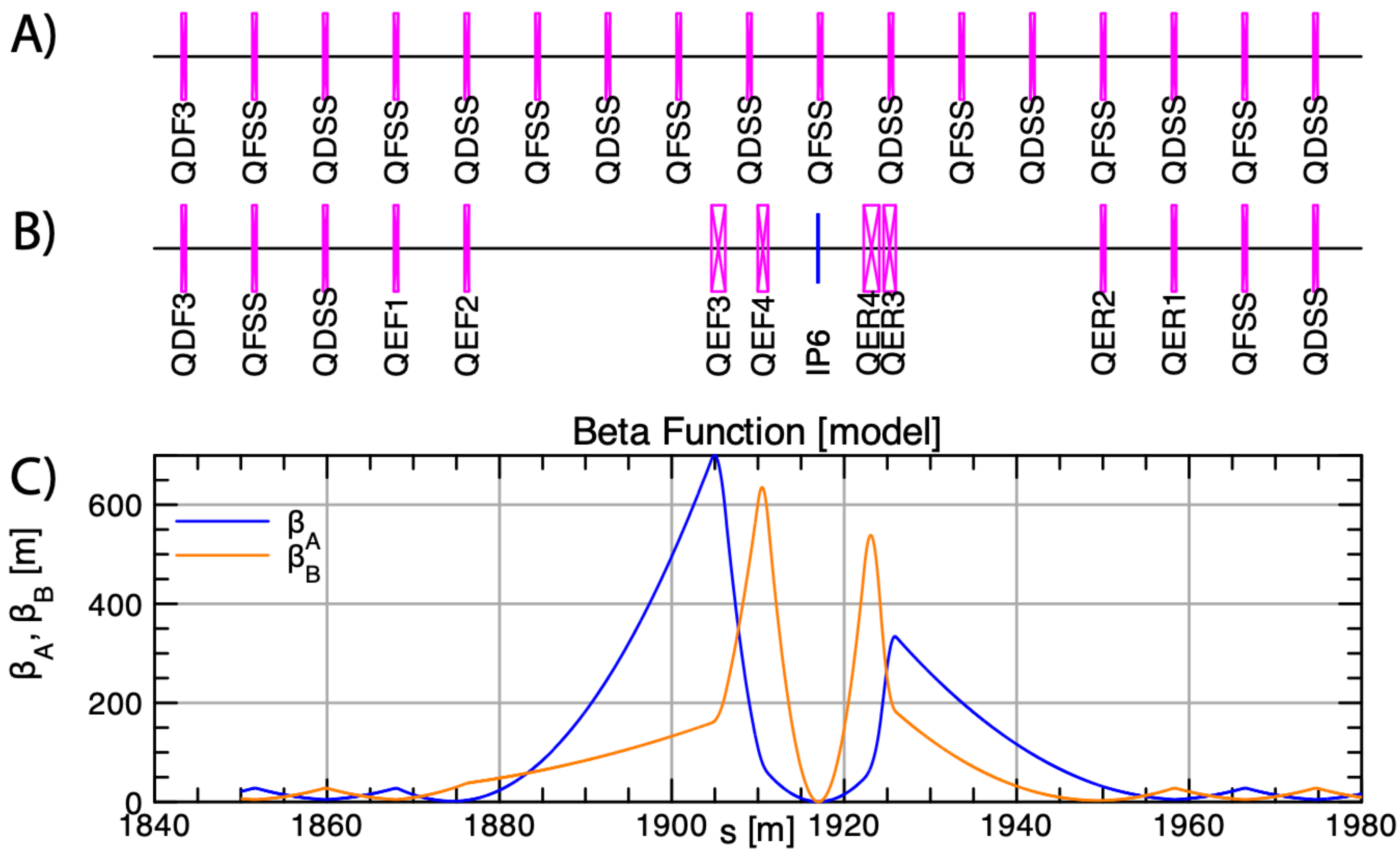
The forward arc with dispersion suppressors



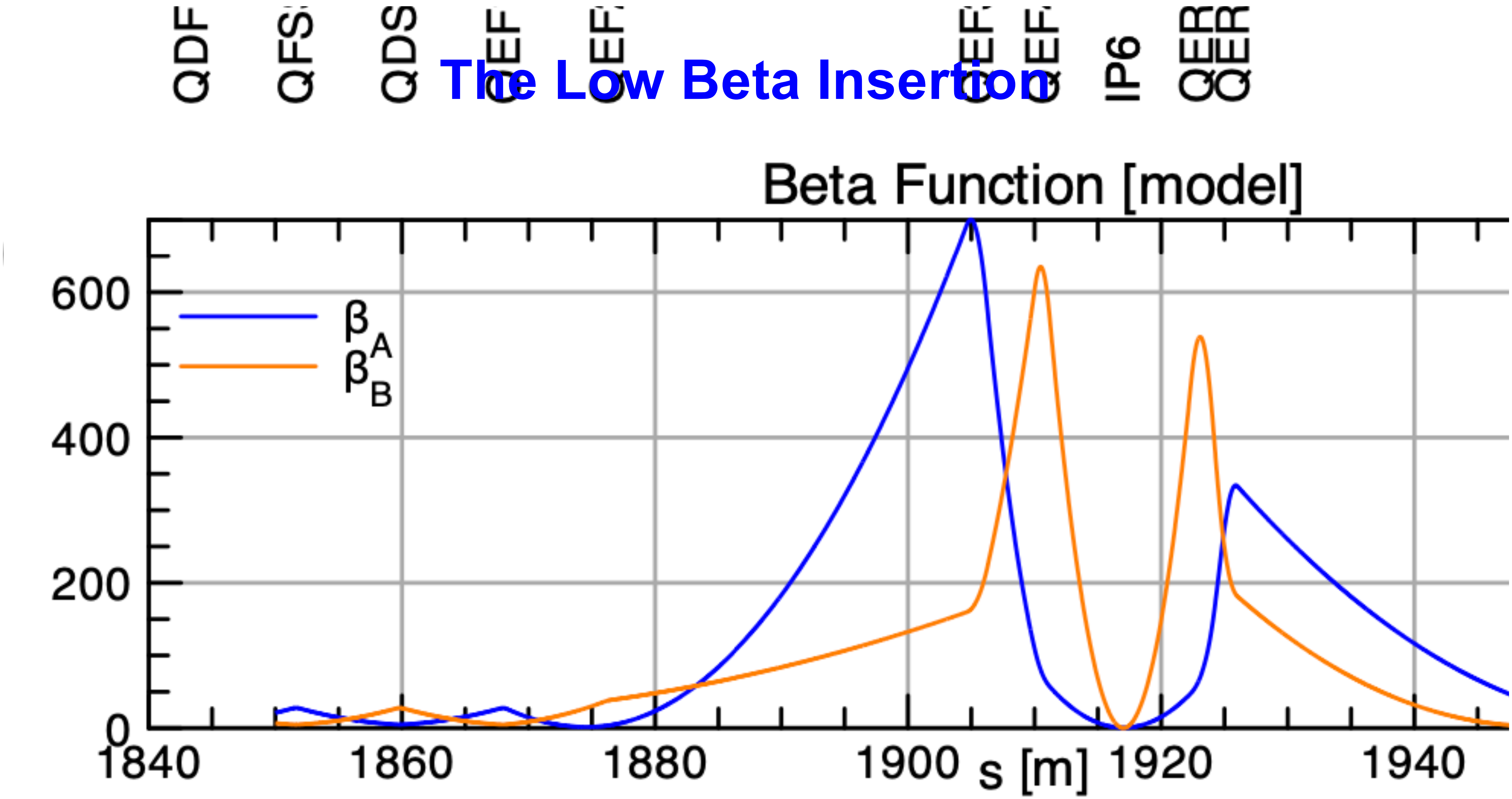
The forward arc with dispersion suppressors



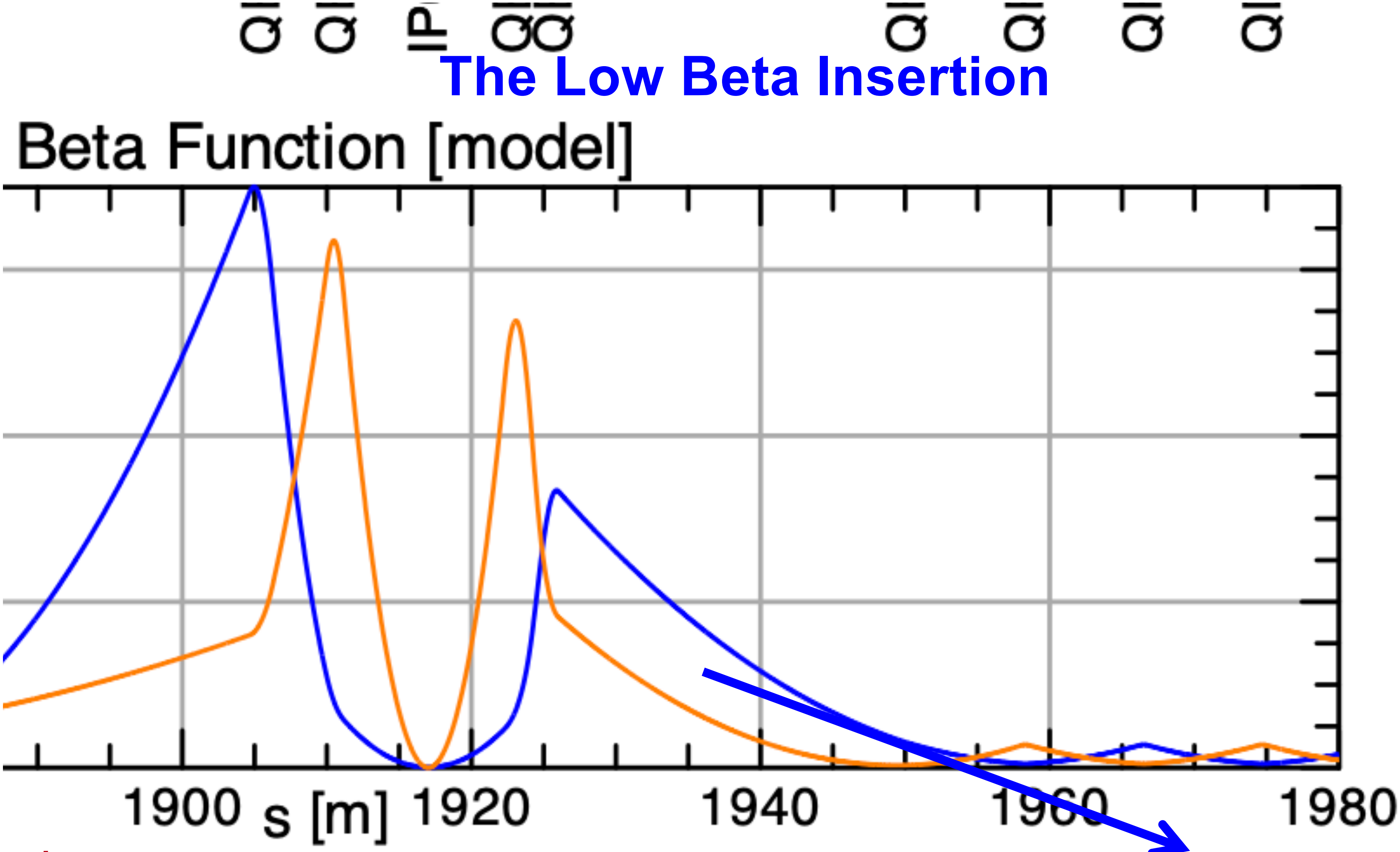
The Low Beta Insertion



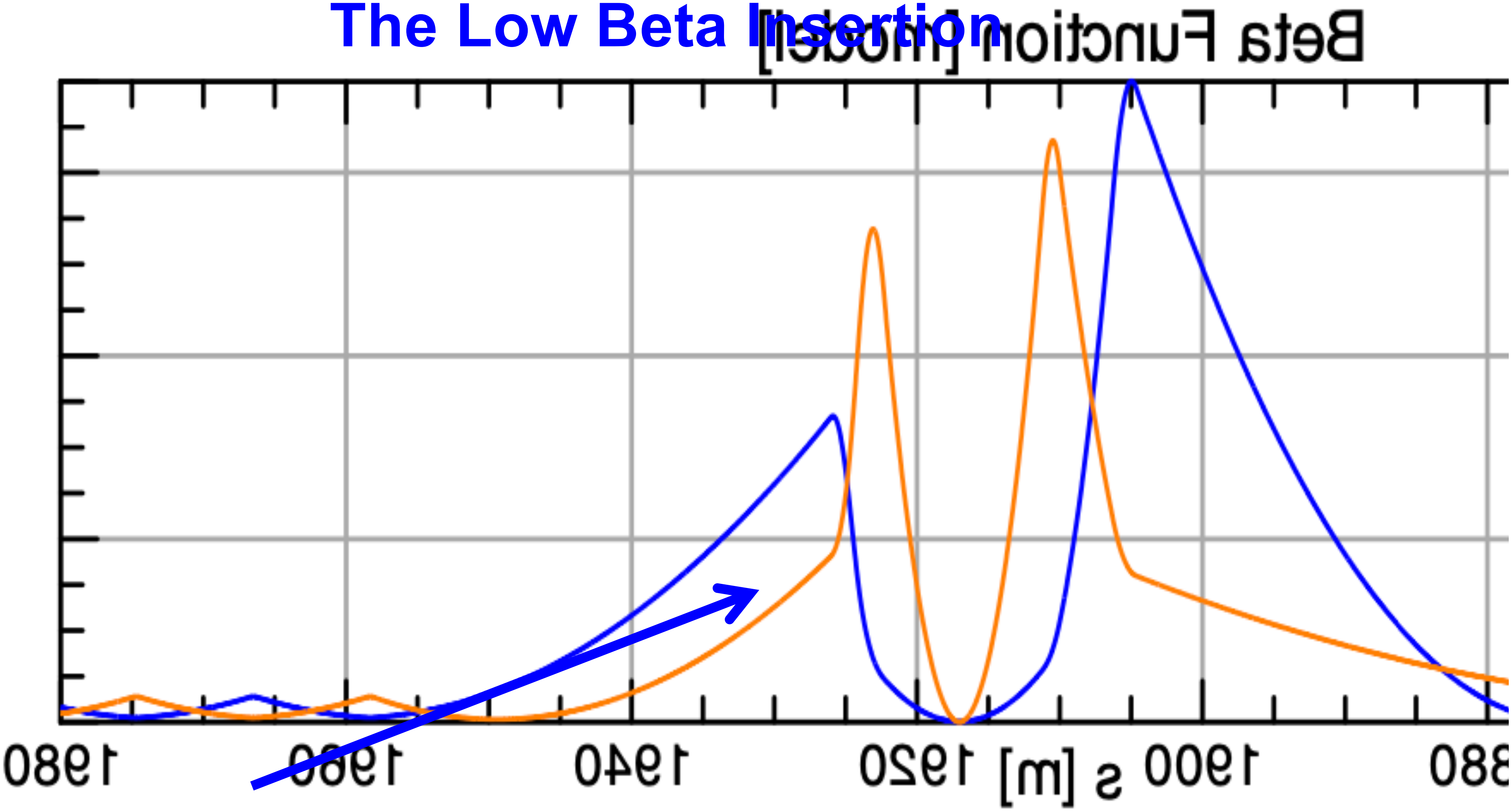
The Low Beta Insertion



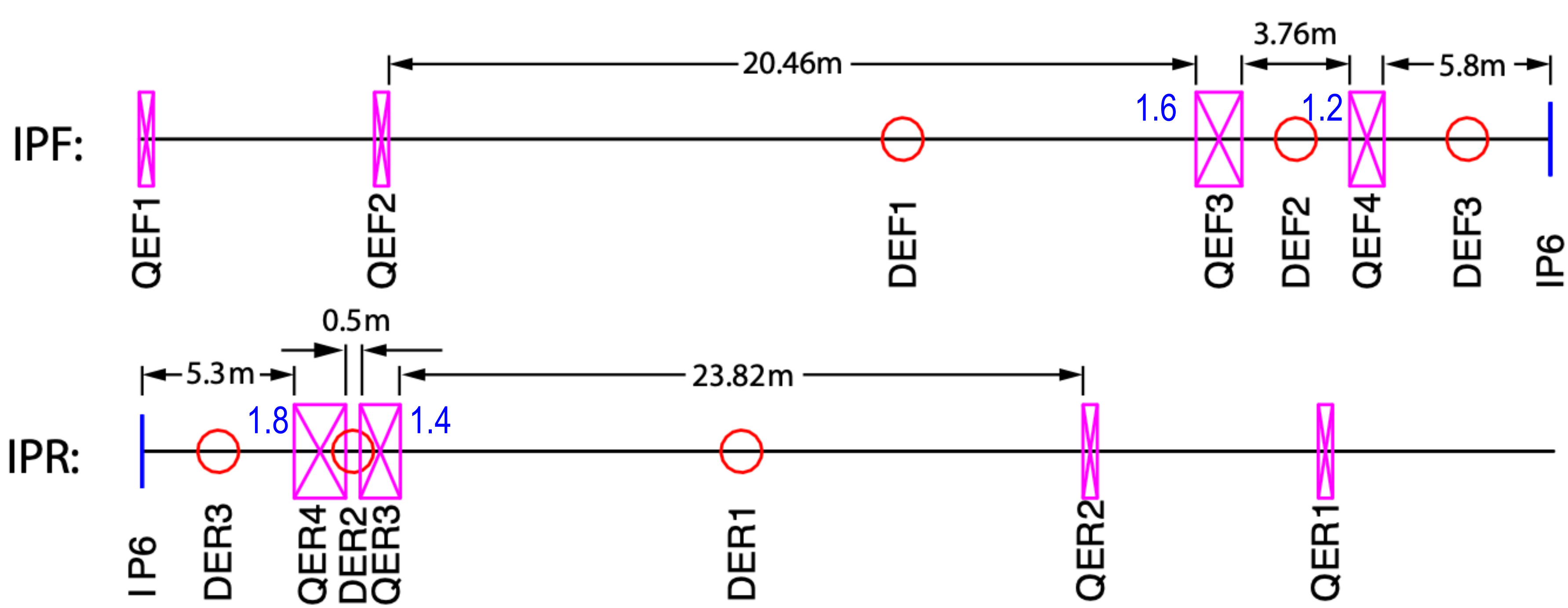
The Low Beta Insertion



The Low Beta Insertion



Dimensions for our exercise



The phase ellipse

Particles with a common J and different ϕ all lie on an ellipse in phase space:

$$\begin{pmatrix} x \\ x' \end{pmatrix} = \sqrt{2J}\begin{pmatrix} \sqrt{\beta} & 0 \\ -\frac{\alpha}{\sqrt{\beta}} & \frac{I}{\sqrt{\beta}} \end{pmatrix}\begin{pmatrix} \sin(\psi(s)+\phi_0) \\ \cos(\psi(s)+\phi_0) \end{pmatrix}$$

(Linear transform of a circle)

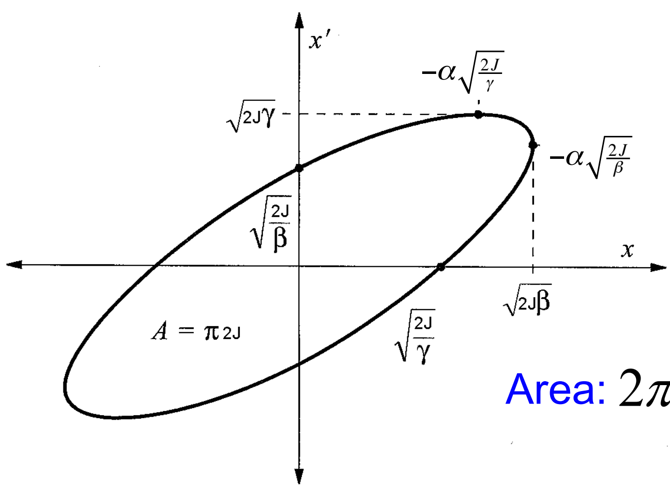
$$x_{\max} = \sqrt{2J\beta} \text{ at } x' = -\alpha\sqrt{\tfrac{2J}{\beta}}$$

$$(x,x')\begin{pmatrix} \frac{I}{\sqrt{\beta}} & \frac{\alpha}{\sqrt{\beta}} \\ 0 & \sqrt{\beta} \end{pmatrix}\begin{pmatrix} \frac{I}{\sqrt{\beta}} & 0 \\ \frac{\alpha}{\sqrt{\beta}} & \sqrt{\beta} \end{pmatrix}\begin{pmatrix} x \\ x' \end{pmatrix} = (x,x')\begin{pmatrix} \gamma & \alpha \\ \alpha & \beta \end{pmatrix}\begin{pmatrix} x \\ x' \end{pmatrix} = 2J$$

(Quadratic form)

$$\beta\gamma - \alpha^2 = I^2$$

Area: $2\pi J / I$



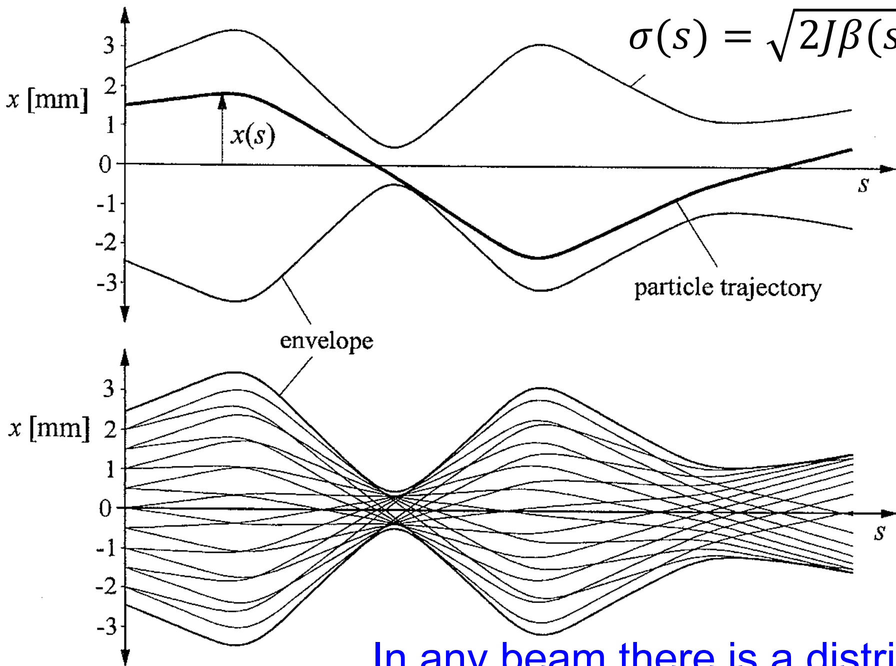
I=1 is therefore a useful choice!

What β is for x, γ is for x'

$$x'_{\max} = \sqrt{2J\gamma} \text{ at } x = -\alpha\sqrt{\tfrac{2J}{\gamma}}$$

Area: $2\pi J$ → $$\int_0^{2\pi}\int_0^{J} dJ\,d\phi = 2\pi J = \iint dx\,dx'$$

The beam envelope



$$\sigma(s) = \sqrt{2J\beta(s)}$$

$$x(s) = \sqrt{2J\beta(s)}\,\sin(\psi(s) + \phi_0)$$

$$\sigma' = -\alpha\sqrt{\frac{2J}{\beta}}$$

$$\sigma'' = -(k\beta - \gamma)\sqrt{\frac{2J}{\beta}} - \alpha^2\sqrt{\frac{2J}{\beta^3}}$$

$$\sigma'' = -(k\beta^2 - 1)\sqrt{\frac{2J}{\beta^3}}$$

In any beam there is a distribution of initial parameters. If the particles with the largest J are distributed in ϕ over all angles, then the envelope of the beam is described by $\sigma = \sqrt{2J_{max}\beta(s)}$.

The initial conditions of β and α are chosen so that this is approximately the case.

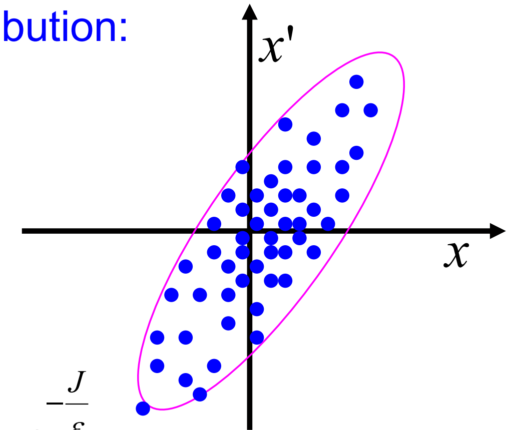
The envelope equation: $\sigma'' = -k\sigma + \frac{(2J)^2}{\sigma^3}$

The phase space distribution

Often one can fit a Gauss distribution to the particle distribution:

$$\rho(x,x') = \frac{1}{2\pi\varepsilon} e^{-\frac{\gamma x^2 + 2\alpha x x' + \beta x'^2}{2\varepsilon}}$$

The equi-density lines are then ellipses. And one chooses the starting conditions for β and α according to these ellipses!



$$\begin{pmatrix} x \\ x' \end{pmatrix} = \sqrt{2J} \begin{pmatrix} \sqrt{\beta} & 0 \\ -\frac{\alpha}{\sqrt{\beta}} & \frac{1}{\sqrt{\beta}} \end{pmatrix} \begin{pmatrix} \sin\phi_0 \\ \cos\phi_0 \end{pmatrix} \qquad \rho(J,\phi_0) = \frac{1}{2\pi\varepsilon} e^{-\frac{J}{\varepsilon}}$$

$$\langle 1 \rangle = \frac{1}{2\pi\varepsilon} \int_0^{2\pi} \int_0^{\infty} e^{-J/\varepsilon} dJ d\phi_0 = 1$$

Initial beam distribution ⟶ initial α, β, γ

$$\langle x^2 \rangle = \frac{1}{2\pi\varepsilon} \iint 2J\beta \sin\phi_0^2 e^{-J/\varepsilon} dJ d\phi_0 = \varepsilon\beta \longrightarrow \langle x'^2 \rangle = \varepsilon\gamma$$

$$\langle x x' \rangle = -\frac{1}{2\pi\varepsilon} \iint 2J\alpha \sin\varphi_0^2 e^{-J/\varepsilon} dJ d\varphi_0 = -\varepsilon\alpha$$

$$\varepsilon = \sqrt{\langle x^2 \rangle \langle x'^2 \rangle - \langle x x' \rangle^2}$$ is called the emittance.

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Invariant of motion

$$x(s) = \sqrt{2J\beta(s)}\sin(\psi(s)+\phi_0)$$

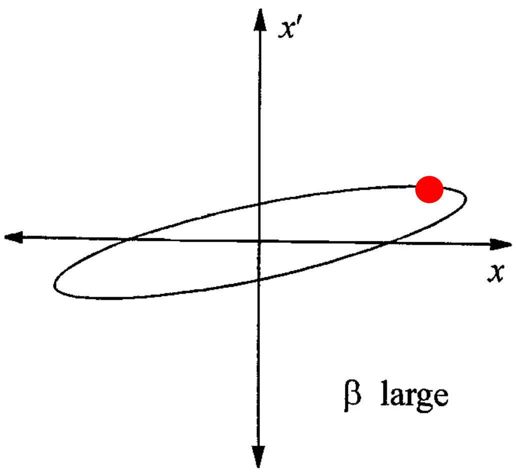
Where J and ϕ are given by the starting conditions x_0 and x'_0.

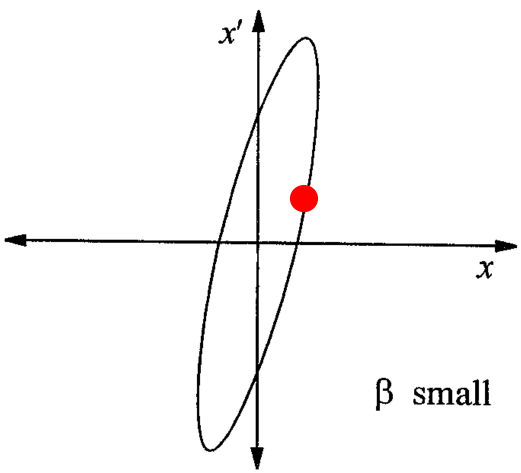
$$\gamma x^2 + 2\alpha x x' + \beta x'^2 = 2J$$

Leads to the invariant of motion:

$$f(x,x',s) = \gamma(s)x^2 + 2\alpha(s)xx' + \beta(s)x'^2 \quad \Rightarrow \quad \tfrac{d}{ds} f = 0$$

It is called the Courant-Snyder invariant.





Twiss differential equation ➔ usually too hard

$$\gamma = \frac{1+\alpha^2}{\beta}$$
$$\beta' = -2\alpha$$
$$\alpha' = k\beta - \gamma$$

$$\beta'' = 2\gamma = 2\frac{1+\frac{1}{4}\beta'^2}{\beta} = \frac{d\beta'}{d\beta}\frac{d\beta}{ds}$$
$$\frac{\beta'}{1+\frac{1}{4}\beta'^2}d\beta' = 2\frac{d\beta}{\beta}$$
$$\log(1+\tfrac{1}{4}\beta'^2) = \log(\beta/\beta_0)$$
$$\beta' = 2\sqrt{\beta/\beta_0 - 1}$$
$$\frac{d\beta}{2\sqrt{\beta/\beta_0 - 1}} = ds$$
$$\beta_0\sqrt{\beta/\beta_0 - 1} = s - s_0$$
$$\beta(s) = \beta_0\left(1+\left(\frac{s-s_0}{\beta_0}\right)^2\right)$$

Propagation of Twiss parameters

$$(x_0, x_0')\begin{pmatrix}\gamma_0 & \alpha_0\\ \alpha_0 & \beta_0\end{pmatrix}\begin{pmatrix}x_0\\ x_0'\end{pmatrix} = 2J$$

$$(x, x')\begin{pmatrix}\gamma & \alpha\\ \alpha & \beta\end{pmatrix}\begin{pmatrix}x\\ x'\end{pmatrix} = 2J = (x_0, x_0')\underline{M}^T\begin{pmatrix}\gamma & \alpha\\ \alpha & \beta\end{pmatrix}\underline{M}\begin{pmatrix}x_0\\ x_0'\end{pmatrix}$$

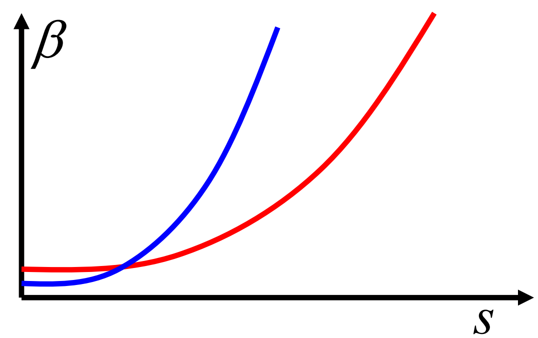
$$\begin{pmatrix}\gamma & \alpha\\ \alpha & \beta\end{pmatrix} = \underline{M}^{-T}\begin{pmatrix}\gamma_0 & \alpha_0\\ \alpha_0 & \beta_0\end{pmatrix}\underline{M}^{-1}$$

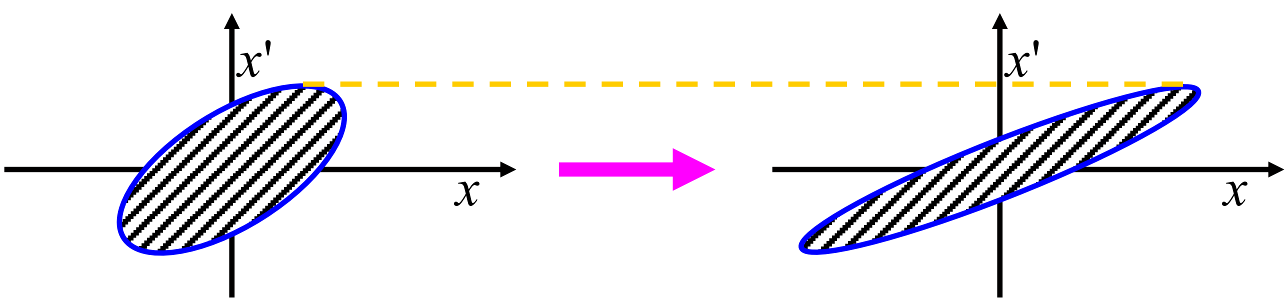
$$\begin{pmatrix}\beta & -\alpha\\ -\alpha & \gamma\end{pmatrix} = \underline{M}\begin{pmatrix}\beta_0 & -\alpha_0\\ -\alpha_0 & \gamma_0\end{pmatrix}\underline{M}^T$$

Twiss parameters in a drift

$$\begin{pmatrix} \beta & -\alpha \\ -\alpha & \gamma \end{pmatrix} = \begin{pmatrix} 1 & s \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \beta_0 & -\alpha_0 \\ -\alpha_0 & \gamma_0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ s & 1 \end{pmatrix} = \begin{pmatrix} \beta_0 - 2\alpha_0 s + \gamma_0 s^2 & \gamma_0 s - \alpha_0 \\ \gamma_0 s - \alpha_0 & \gamma_0 \end{pmatrix}$$

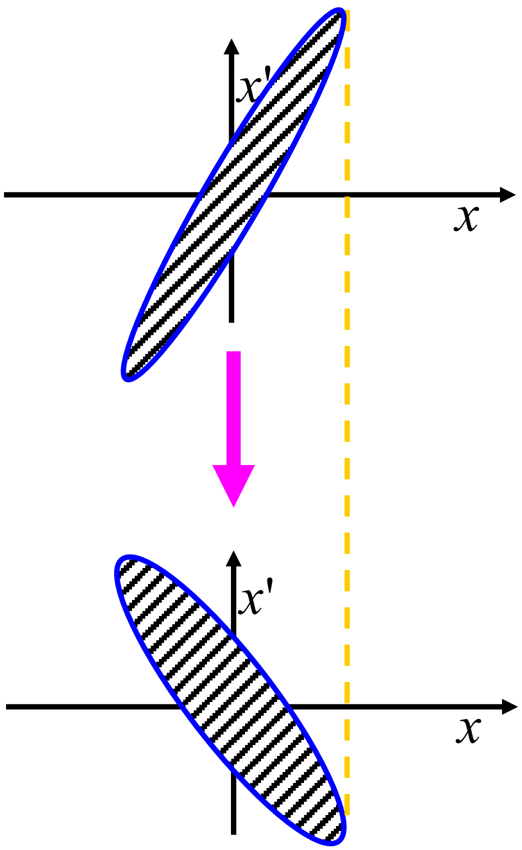
$$\beta = \beta_0^* [1 + (\tfrac{s}{\beta_0^*})^2] \quad \text{for} \quad \alpha_0^* = 0$$



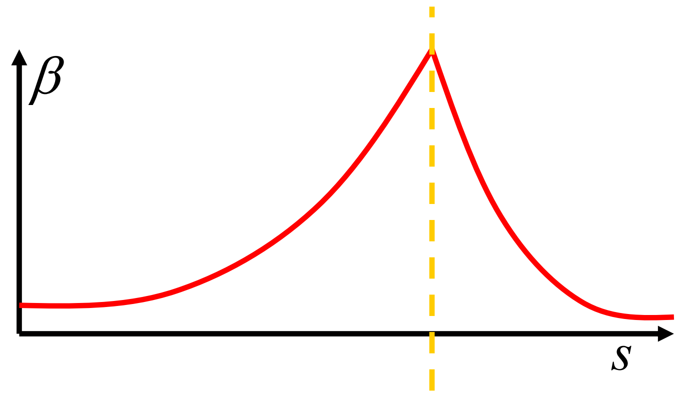


Twiss parameters in a thin quadrupole

$$\begin{pmatrix} \beta & -\alpha \\ -\alpha & \gamma \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ -k & 1 \end{pmatrix} \begin{pmatrix} \beta_0 & -\alpha_0 \\ -\alpha_0 & \gamma_0 \end{pmatrix} \begin{pmatrix} 1 & -k \\ 0 & 1 \end{pmatrix}$$



$$\alpha = \alpha_0 + k\beta_0$$



From Twiss parameter to Transfer Marix

$$\begin{pmatrix} x_0 \\ x_0' \end{pmatrix} = \sqrt{2J} \begin{pmatrix} \sqrt{\beta_0} & 0 \\ -\frac{\alpha_0}{\sqrt{\beta_0}} & \frac{1}{\sqrt{\beta_0}} \end{pmatrix} \begin{pmatrix} \sin(\phi_0) \\ \cos(\phi_0) \end{pmatrix}$$

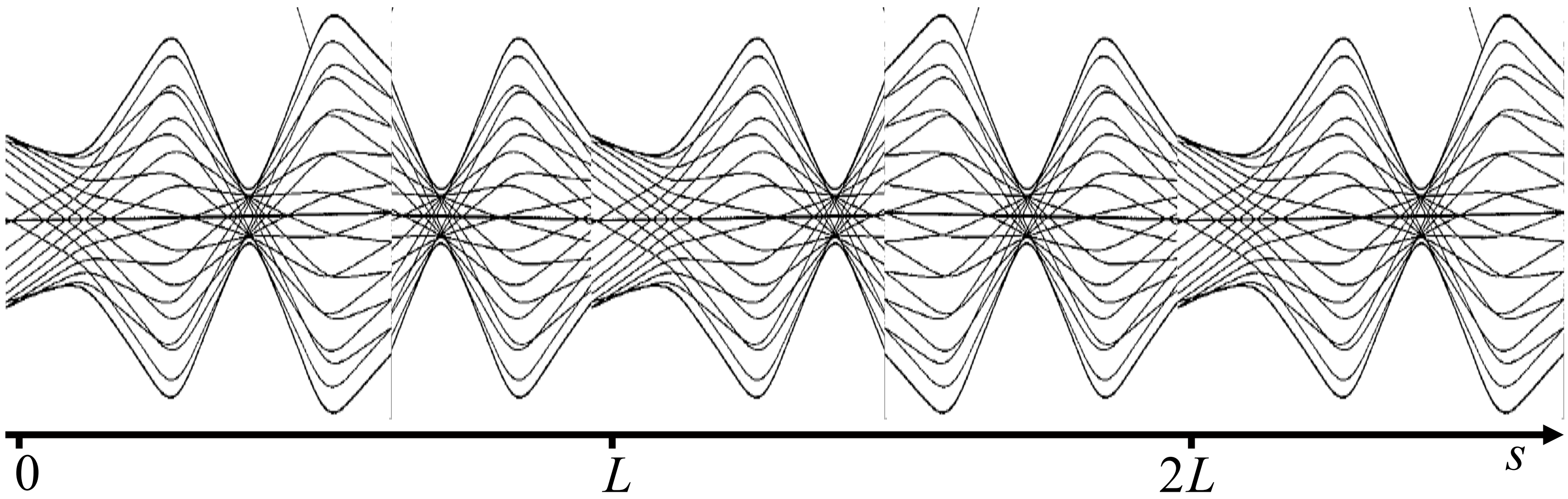
$$\begin{pmatrix} x \\ x' \end{pmatrix} = \sqrt{2J} \begin{pmatrix} \sqrt{\beta} & 0 \\ -\frac{\alpha}{\sqrt{\beta}} & \frac{1}{\sqrt{\beta}} \end{pmatrix} \begin{pmatrix} \sin(\psi(s)+\phi_0) \\ \cos(\psi(s)+\phi_0) \end{pmatrix}$$

$$= \sqrt{2J} \begin{pmatrix} \sqrt{\beta} & 0 \\ -\frac{\alpha}{\sqrt{\beta}} & \frac{1}{\sqrt{\beta}} \end{pmatrix} \begin{pmatrix} \cos\psi(s) & \sin\psi(s) \\ -\sin\psi(s) & \cos\psi(s) \end{pmatrix} \begin{pmatrix} \sin\phi_0 \\ \cos\phi_0 \end{pmatrix}$$

$$\underline{M}(s) = \begin{pmatrix} \sqrt{\beta} & 0 \\ -\frac{\alpha}{\sqrt{\beta}} & \frac{1}{\sqrt{\beta}} \end{pmatrix} \begin{pmatrix} \cos\psi(s) & \sin\psi(s) \\ -\sin\psi(s) & \cos\psi(s) \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{\beta_0}} & 0 \\ \frac{\alpha_0}{\sqrt{\beta_0}} & \sqrt{\beta_0} \end{pmatrix}$$

$$= \begin{pmatrix} \sqrt{\frac{\beta}{\beta_0}}[\cos\psi + \alpha_0 \sin\psi] & \sqrt{\beta_0 \beta} \sin\psi \\ \sqrt{\frac{1}{\beta_0 \beta}}[(\alpha_0 - \alpha)\cos\psi - (1+\alpha_0 \alpha)\sin\psi] & \sqrt{\frac{\beta_0}{\beta}}[\cos\psi - \alpha \sin\psi] \end{pmatrix}$$

Periodic solutions in a periodic accelerator



$$\vec{z}(s) = \underline{M}(s,0)\vec{z}(0)$$

$$\vec{z}(L) = \underline{M}(L,0)\vec{z}(0)$$

$$\vec{z}(s+L) = \underline{M}_0(s)\vec{z}(s) \quad , \quad \underline{M}_0 = \underline{M}(s+L,s)$$

$$\vec{z}(s+nL) = \underline{M}_0^n(s)\vec{z}(s)$$

Periodic beta functions

If the particle distribution in a ring or any other periodic structure is stable, it is periodic from turn to turn.

$$\rho(x,x',s+L)=\rho(x,x',s)$$

To be matched to such a beam, the Twiss parameters α, β, γ must be the same after every turn.

$$\underline{M}(s,0)=\begin{pmatrix}\sqrt{\frac{\beta}{\beta_0}}[\cos\psi+\alpha_0\sin\psi] & \sqrt{\beta_0\beta}\sin\psi\\ \sqrt{\frac{1}{\beta_0\beta}}[(\alpha_0-\alpha)\cos\psi-(1+\alpha_0\alpha)\sin\psi] & \sqrt{\frac{\beta_0}{\beta}}[\cos\psi-\alpha\sin\psi]\end{pmatrix}$$

$$\underline{M}_p(s)=\begin{pmatrix}\cos\mu+\alpha\sin\mu & \beta\sin\mu\\ -\gamma\sin\mu & \cos\mu-\alpha\sin\mu\end{pmatrix}=\underline{1}\cos\mu+\begin{pmatrix}\alpha & \beta\\ -\gamma & -\alpha\end{pmatrix}\sin\mu$$

$$\mu=\psi(s+L)-\psi(s)$$

One turn matrix to periodic Twiss parameters

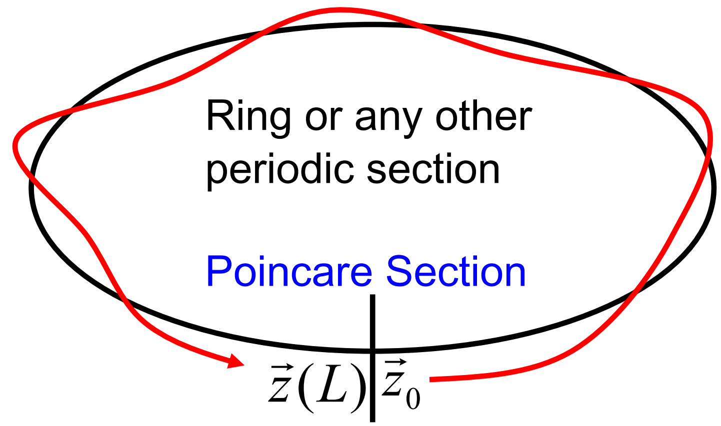
The periodic Twiss parameters are the solution of a nonlinear differential equation with periodic boundary conditions:

$$\beta' = -2\alpha \qquad \text{with} \qquad \beta(L) = \beta(0)$$

$$\alpha' = k\beta - \tfrac{1+\alpha^2}{\beta} \qquad \text{with} \qquad \alpha(L) = \alpha(0)$$

$$\mu = \int_0^L \tfrac{1}{\beta(\hat{s})}\, d\hat{s}$$

Note: $\beta(s) > 0$



$$\underline{M}_0(s) = \underline{1}\cos\mu + \underline{\beta}\sin\mu \,;\, \underline{\beta} = \begin{pmatrix} \alpha & \beta \\ -\gamma & -\alpha \end{pmatrix}$$

$$\cos\mu = \tfrac{1}{2}\,\mathrm{Tr}[\underline{M}_0(s)]$$

$$\beta = \underline{M}_{0,12}\,\tfrac{1}{\sin\mu}$$

$$\alpha = (\underline{M}_{0,11} - \underline{M}_{0,22})\,\tfrac{1}{2\sin\mu}$$

$$\gamma = \tfrac{1+\alpha^2}{\beta}$$

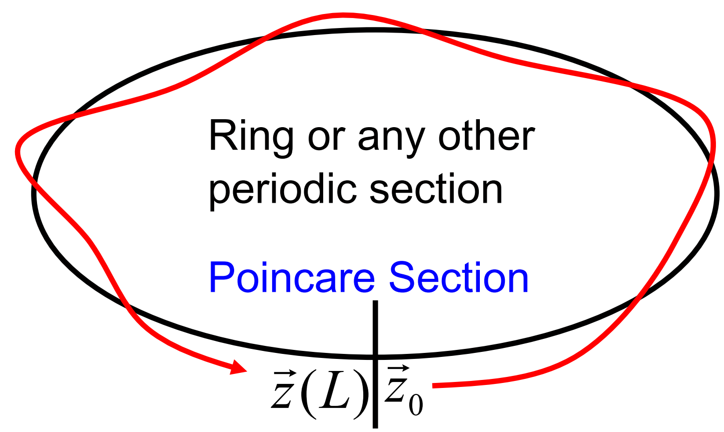
Stable beam motion and thus a periodic beta function can only exist when |Tr[M]| < 2.

The tune of a particle accelerator

The betatron phase advance per turn devided by 2π is called the TUNE.

$$\mu = 2\pi\nu = \psi(s+L) - \psi(s)$$

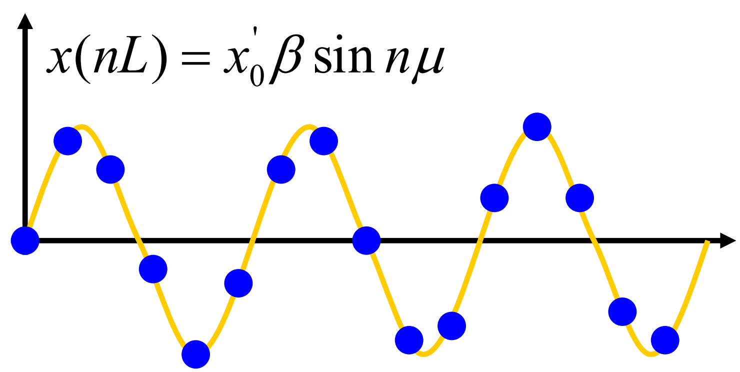
It is a property of the ring and does not depend on the azimuth s.



$$\underline{M}_0(s) = \underline{1}\cos\mu + \begin{pmatrix} \alpha(s) & \beta(s) \\ -\gamma(s) & -\alpha(s) \end{pmatrix} \sin\mu$$

$$2\cos\mu(s) = \mathrm{Tr}[\underline{M}_0(s)] = \mathrm{Tr}[\underline{M}(s,0)\underline{M}_0(0)\underline{M}^{-1}(s,0)]$$

$$= \mathrm{Tr}[\underline{M}_0(0)] = 2\cos\mu(0)$$



$$\underline{M}_0^n = \underline{1}\cos n\mu + \underline{\beta}\sin n\mu$$

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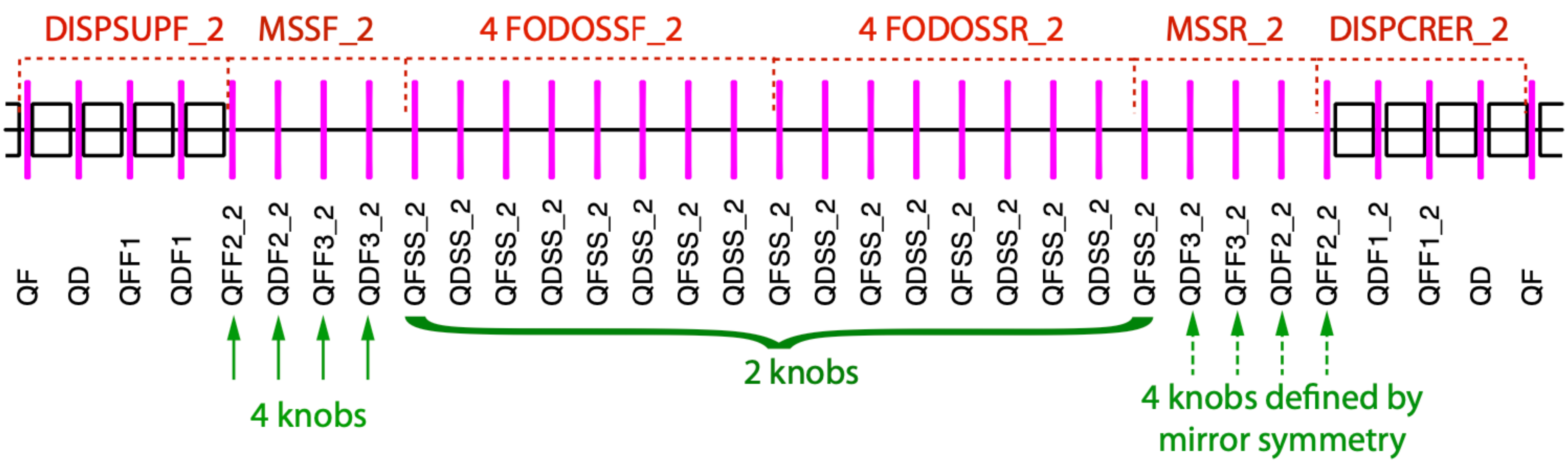
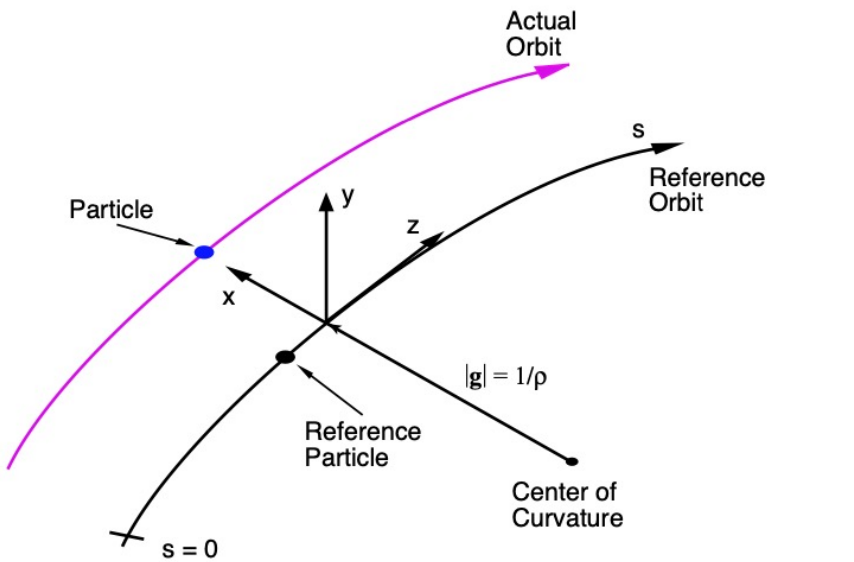
Changing the tune without beta beat

Changing quadrupoles changes
the beam envelope ➔ the beta function ➔ the phase advance ➔ the TUNE

To change the tune without creating a beta beat around the ring we need to install a tune matching section.



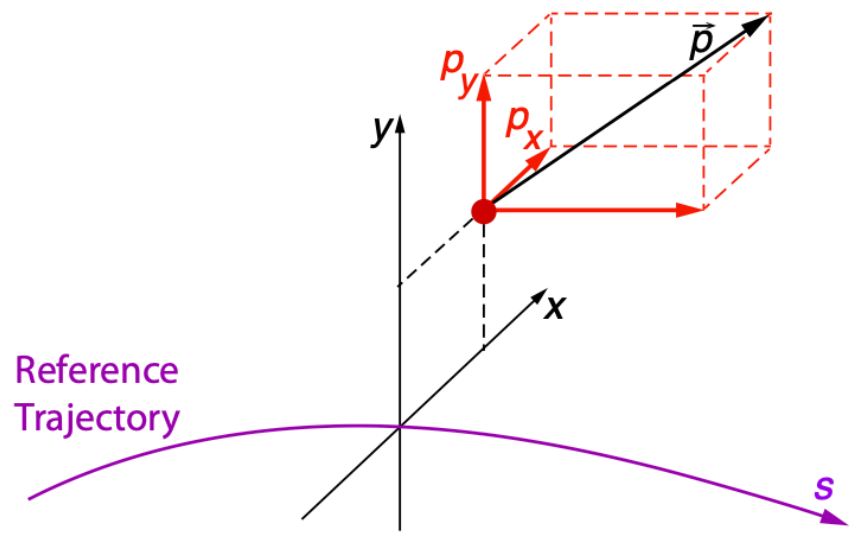
Particle coordinates in Bmad



(a) Particle coordinate positions are relative to the reference orbit.

Bmad coordinates:

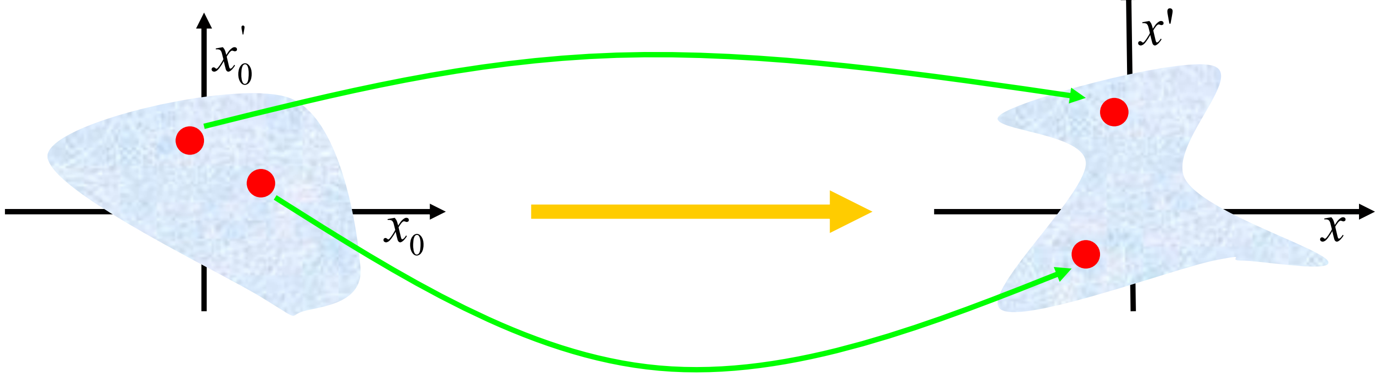
```
px  =  Px  /  P0
py  =  Py  /  P0
pz  =  (P  –  P0)  /  P0
z   =  c * β * (t_ref – t)
```



(b) Particle phase space.

Liouville's Theorem

- A phase space volume does not change when it is transported by Hamiltonian motion. $\vec{z}(s) = \underline{M}(s) \cdot \vec{z}_0$ with $\det[\underline{M}(s)] = +1$



Volume = $V = \iint_V d^n\vec{z} = \iint_{V_0} \left|\frac{\partial\vec{z}}{\partial\vec{z}_0}\right| d^n\vec{z}_0 = \iint_{V_0} |\underline{M}| d^n\vec{z}_0 = \iint_{V_0} d^n\vec{z}_0 = V_0$

Hamiltonian Motion → $V = V_0$

But Hamiltonian requires symplecticity, which is much more than just $\det[\underline{M}(s)] = +1$

The 4-dimensional equation of motion

$$\frac{d^2}{dt^2}\vec{r} = \vec{f}_r(\vec{r}, \tfrac{d}{dt}\vec{r}, t)$$

3 dimensional ODE of 2nd order can be changed to a

6 dimensional ODE of 1st order:

$$\left.\begin{aligned} \frac{d}{dt}\vec{r} &= \frac{1}{m\gamma}\vec{p} = \frac{c}{\sqrt{p^2-(mc)^2}}\vec{p} \\ \frac{d}{dt}\vec{p} &= \vec{F}(\vec{r},\vec{p},t) \end{aligned}\right\} \quad \frac{d}{dt}\vec{Z} = \vec{f}_Z(\vec{Z},t)\,, \quad \vec{Z} = (\vec{r},\vec{p})$$

If the force does not depend on time, as in a typical beam line magnet, the energy is conserved so that one can reduce the dimension to 5. The equation of motion is then autonomous.

Furthermore, the time dependence is often not as interesting as the trajectory along the accelerator length “s”. Using “s” as the independent variable reduces the dimensions to 4. The equation of motion is then no longer autonomous.

$$\frac{d}{ds}\vec{z} = \vec{f}_z(\vec{z},s)\,, \quad \vec{z} = (x, y, p_x, p_y)$$

6D equation of motion

Usually one prefers to compute the trajectory as a function of "s" along the accelerator even when the energy is not conserved, as when accelerating cavities are in the accelerator.

Then the energy "E" and the time "t" at which a particle arrives at the cavities are important. And the equations become 6 dimensional again:

$$\tfrac{d}{ds}\vec{z} = \vec{f}_z(\vec{z},s)\,, \quad \vec{z} = (x,y,p_x,p_y,-t,E)$$

But: $\vec{z} = (\vec{r},\vec{p})$ is an especially suitable variable, since it is a phase space vector so that its equation of motion comes from a Hamiltonian, or by variation principle from a Lagrangian.

$$\delta\int\left[p_x\dot{x} + p_y\dot{y} + p_s\dot{s} - H(\vec{r},\vec{p},t)\right]dt = 0 \quad \Rightarrow \quad \text{Hamiltonian motion}$$

$$\delta\int\left[p_x x' + p_y y' - H\,t' + p_s(x,y,p_x,p_y,t,H)\right]ds = 0 \quad \Rightarrow \quad \text{Hamiltonian motion}$$

The new canonical coordinates are: $\vec{z} = (x,y,p_x,p_y,-t,E) \quad \text{with} \quad \mathrm{E} = \mathrm{H}$

The new Hamiltonian is: $K = -p_s(\vec{z},s)$

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Canonical transformations

If $(x, P_x, y, P_y, -t, E)$ are canonical coordinates to the Hamiltonian K, then

$\left(x, p_x = \frac{P_x}{p_0}, y, p_y = \frac{P_y}{p_0}, -t, \frac{E}{p_0}\right)$ are canonical coordinates to the Hamiltonian $\frac{K_s}{p_0}$.

But Bmad uses the coordinates $\left(x, p_x, y, p_y, z = c\beta(t_0 - t), p_z = \frac{p - p_0}{p_0}\right)$. Are these canonical coordinates?

Bmad's pair z, p_z is also canonical, because it can be related to $-t, \frac{E}{p_0}$ by a symplectic transformation.

$$\mathrm{Det}\begin{pmatrix} \frac{\partial z}{\partial -t} & \frac{\partial z}{\partial E/p_0} \\ \frac{\partial p_z}{\partial -t} & \frac{\partial p_z}{\partial E/p_0} \end{pmatrix} = Det\begin{pmatrix} c\beta & \dots \\ 0 & \frac{\partial p}{\partial E} \end{pmatrix} = 1$$

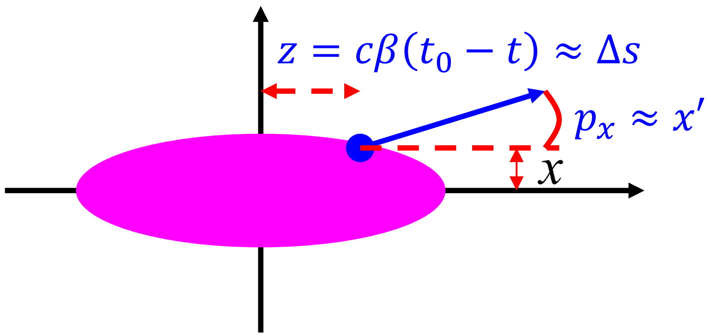
because $p = \frac{1}{c}\sqrt{E^2 - (m_0 c^2)}$ ➔ $\frac{\partial p}{\partial E} = \frac{E}{c^2 p} = \frac{1}{c\beta}$ and all 2x2 matrices with Det=1 are symplectc. Yey ☺

6D phase space motion

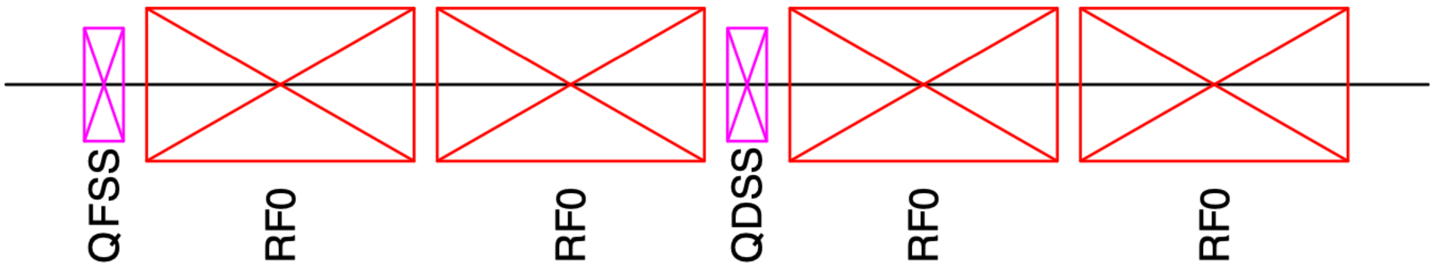
Using a reference momentum p_0 and a reference time t_0:

Bmad's coordinats $\left(x, p_x, y, p_y, z = c\beta(t_0 - t), p_z = \frac{p - p_0}{p_0}\right)$ are canonical, their motion comes from a Hamiltonian is is therefore symplectic.

Usually, p_0 is the design momentum of the beam
And t_0 is the time at which the bunch center is at "s".

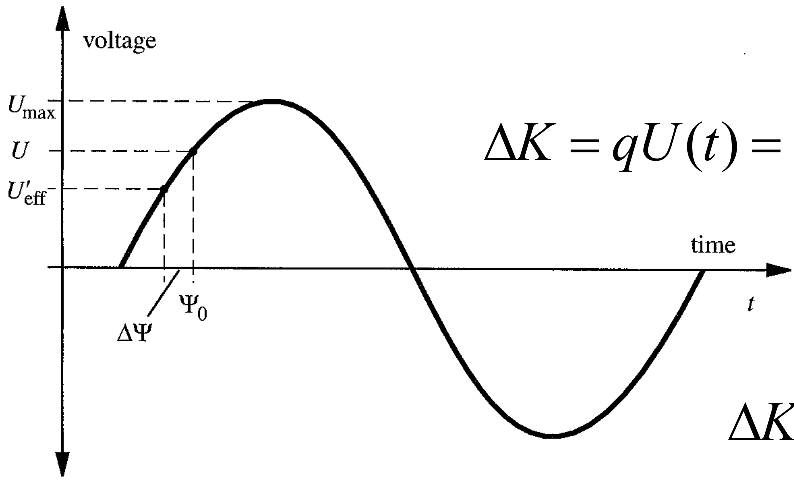


ESR like cavities in the ring



Phase focusing

- 1945: Veksler (UDSSR) and McMillan (USA) realize the importance of phase focusing



$$\Delta K = qU(t) = qU_{\max}\sin(\omega(t-t_0)+\psi_0)$$

Longitudinal position in the bunch:

$$\sigma = s - s_0 = -v_0(t-t_0)$$

$$\Delta K(\sigma) = qU_{\max}\sin(-\tfrac{\omega}{v_0}(s-s_0)+\psi_0)$$

$\Delta K(0) > 0$ (Acceleration)

$\Delta K(\sigma) < \Delta K(0)$ for $\sigma > 0 \Rightarrow \tfrac{d}{d\sigma}\Delta K(\sigma) < 0$ (Phase focusing)

$$\left.\begin{matrix} qU(t) > 0 \\ q\tfrac{d}{dt}U(t) > 0 \end{matrix}\right\} \quad \psi_0 \in (0, \frac{\pi}{2})$$

Phase focusing is required in any RF accelerator.

Longitudinal phase space in a Linac

Other particles: $$\frac{dE}{ds} = \hat{E}\cos\Phi$$

Reference particle: $$\frac{dE_0}{ds} = \hat{E}\cos\Phi_0$$

$$\phi = \Phi - \Phi_0 = \omega(t - t_0)$$

$$\frac{d\delta}{ds} = \tfrac{\hat{E}}{E_0}\left(\cos(\Phi_0 + \phi) - \cos\Phi_0\right) \approx -\phi\tfrac{\hat{E}}{E_0}\sin\Phi_0$$

$$\frac{d\phi}{ds} = \omega\left(\frac{1}{v} - \frac{1}{v_0}\right) \approx \omega\left(\frac{1}{v_0 + \frac{dv}{d\delta}\big|_0 \delta} - \frac{1}{v_0}\right) \approx -\omega\frac{\frac{dv}{d\delta}\big|_0}{v_0^2}\delta = -\omega\frac{c^2}{v_0^3\gamma_0^2}\delta$$

$$\frac{dv}{d\delta} = E\frac{dv}{dE} = \gamma\frac{dv}{d\gamma} = c\gamma\frac{d\sqrt{1 - \frac{1}{\gamma^2}}}{d\gamma} = \frac{c}{\gamma^2\beta}$$

$$\frac{d^2\phi}{ds^2} \approx -\omega\frac{c^2}{v_0^3\gamma_0^2}\frac{d\delta}{ds} \approx \tfrac{\hat{E}}{E_0}\sin\Phi_0\,\omega\frac{c^2}{v_0^3\gamma_0^2}\phi$$

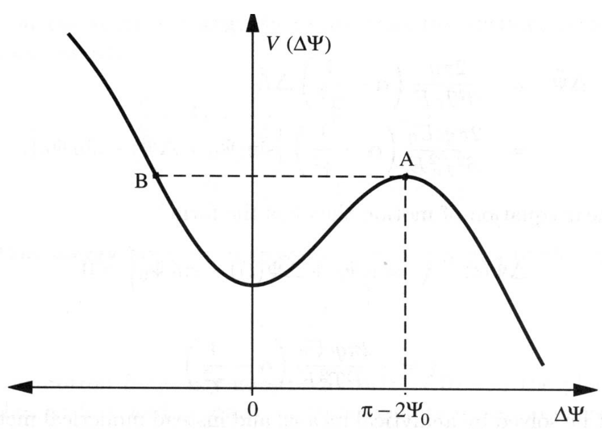
Stability for small phases when the factor on the right-hand side is negative.

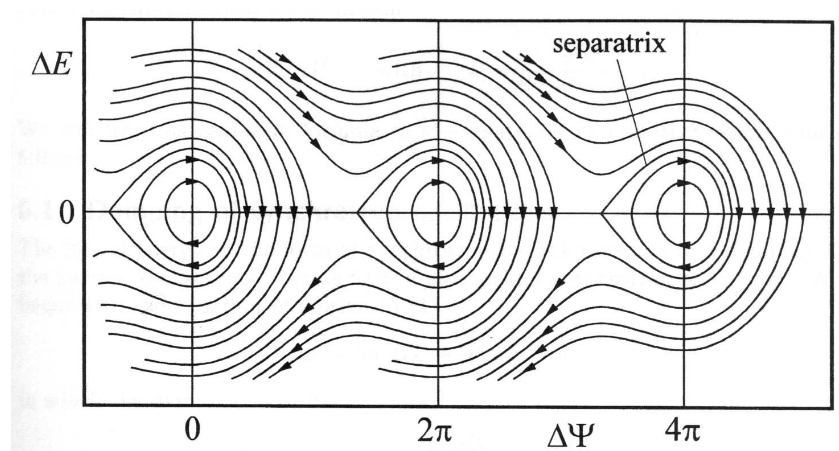
For not very small phases one cannot linearize.

$$\frac{d\delta}{ds} = \tfrac{\hat{E}}{E_0}\left(\cos(\Phi_0 + \phi) - \cos\Phi_0\right) \qquad \frac{d\phi}{ds} \approx -\omega \frac{c^2}{v_0^3 \gamma_0^2}\delta$$

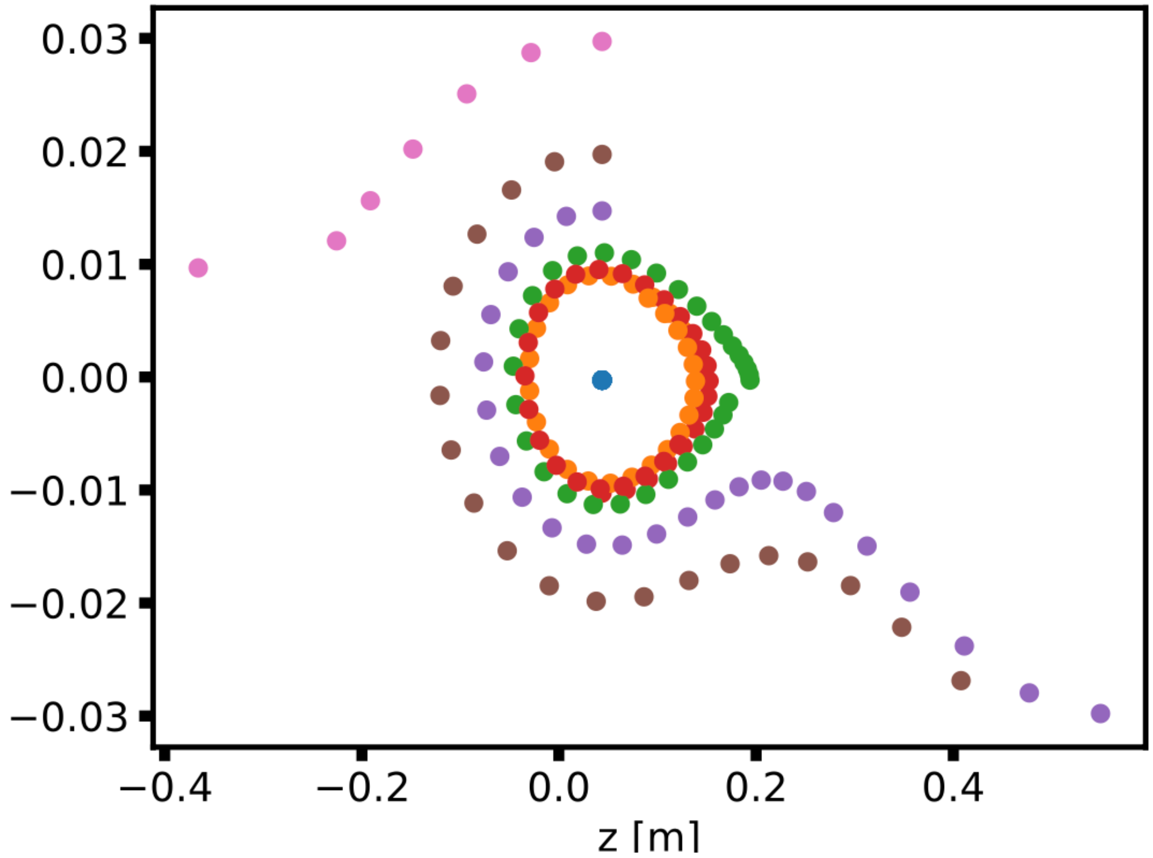
$$H(\phi,\delta) = -\tfrac{q\overline{E_s}}{K_0}\left(\sin(\Phi_0 + \phi) - \phi\cos\Phi_0\right) - \omega\frac{c^2}{v_0^3\gamma_0^2}\delta^2$$

$$\tfrac{d}{dt}\phi = \tfrac{\partial}{\partial\delta}H\ , \quad \tfrac{d}{dt}\delta = -\tfrac{\partial}{\partial\phi}H$$



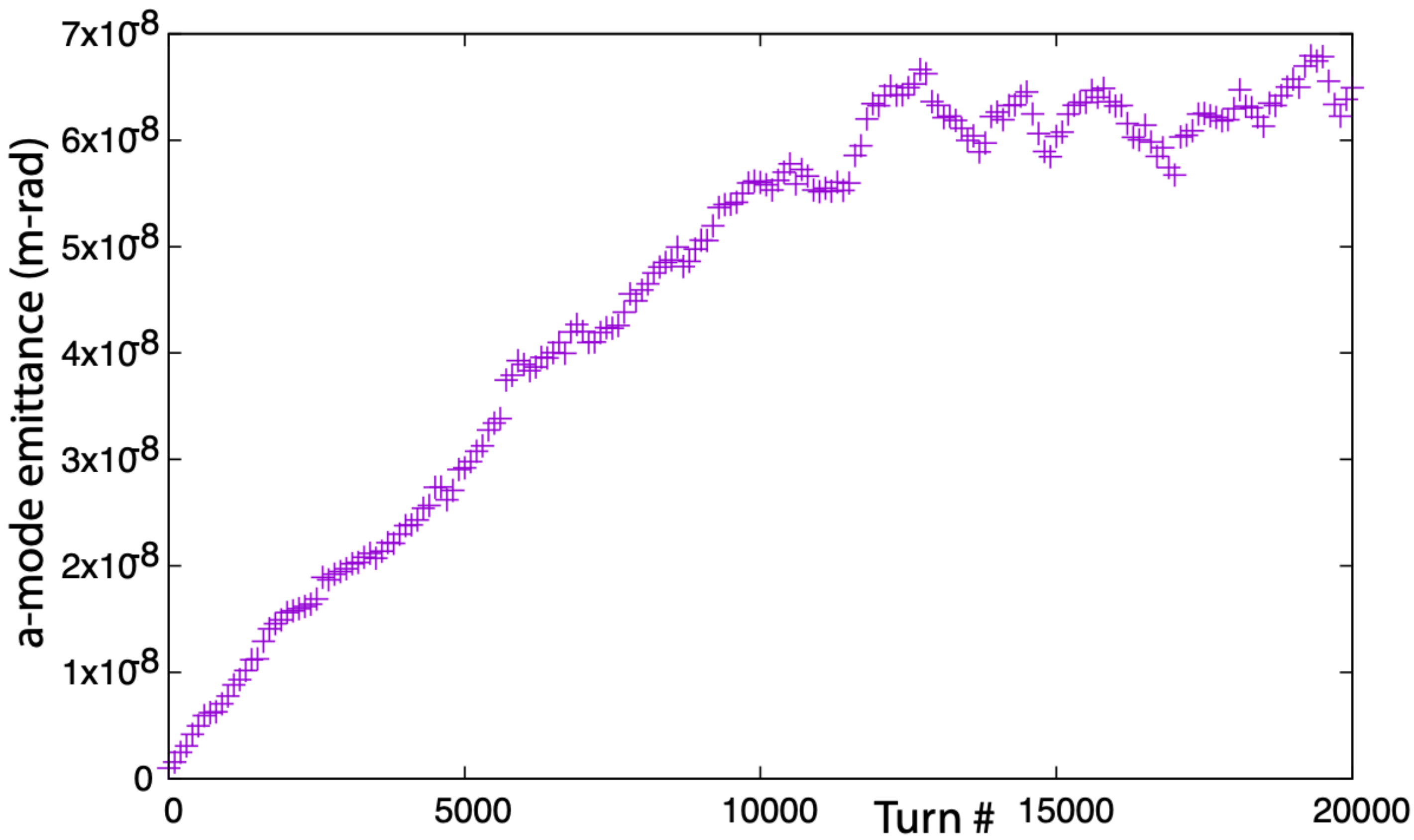


Longitudinal phase space tracking



The trajectory of 7 particles in the z-pz plane over 30 turns.

Transverse phase space tracking



The a-mode emittance as a function of turn number.

Orbit distortions for a one-pass accelerator

$$x' = a$$

$$a' = -(\kappa^2 + k)x + \Delta f$$

The extra force can for example come from an erroneous dipole field or from a correction coil: $\Delta f = \frac{q}{p}\Delta B_y = \Delta\kappa$

Variation of constants: $\vec{z} = \underline{M}\vec{z}_0 + \Delta\vec{z}$ with $\Delta\vec{z} = \int_0^s \underline{M}(s-\hat{s})\begin{pmatrix} 0 \\ \Delta\kappa(\hat{s}) \end{pmatrix} d\hat{s}$

The trajectory of 7 particles in the z-pz plane over 30 turns.

$$\Delta x(s) = \sum_k \Delta\vartheta_k \sqrt{\beta(s)\beta_k}\sin(\psi(s) - \psi_k)$$

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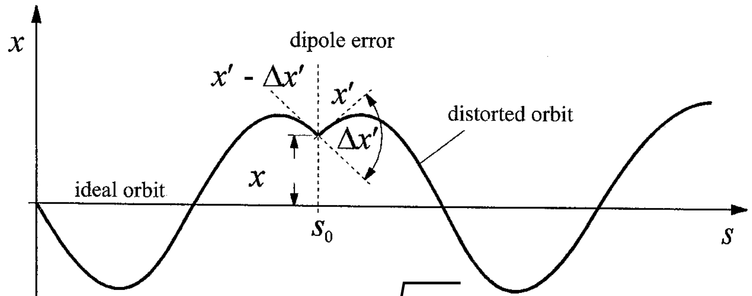
Closed orbit for one kick



$$x_{co}(s_k) = \Delta\vartheta_k \frac{\sqrt{\beta_k}}{2\tan\frac{\mu}{2}}$$

$$x_{\text{co}}(s) = \Delta\vartheta_k \frac{\sqrt{\beta\beta_k}}{2\sin\frac{\mu}{2}} \cos(|\psi - \psi_k| - \tfrac{\mu}{2})$$

Free betatron oscillation

$$x'_{\text{co}}(s_k) - x'_{\text{co}}(s_k + L) = \Delta\vartheta_k \frac{-\sin(-\frac{\mu}{2}) + \sin(\frac{\mu}{2})}{2\sin\frac{\mu}{2}} = \Delta\vartheta_k$$

$$x_{max} = \sqrt{2J\beta}$$

$$x_{\text{co}}(s) = \sqrt{2J\beta}\,\sin(\psi + \varphi_0)\,, \quad J = \frac{\Delta\vartheta_k^2 \beta_k}{8\sin^2\frac{\mu}{2}}$$

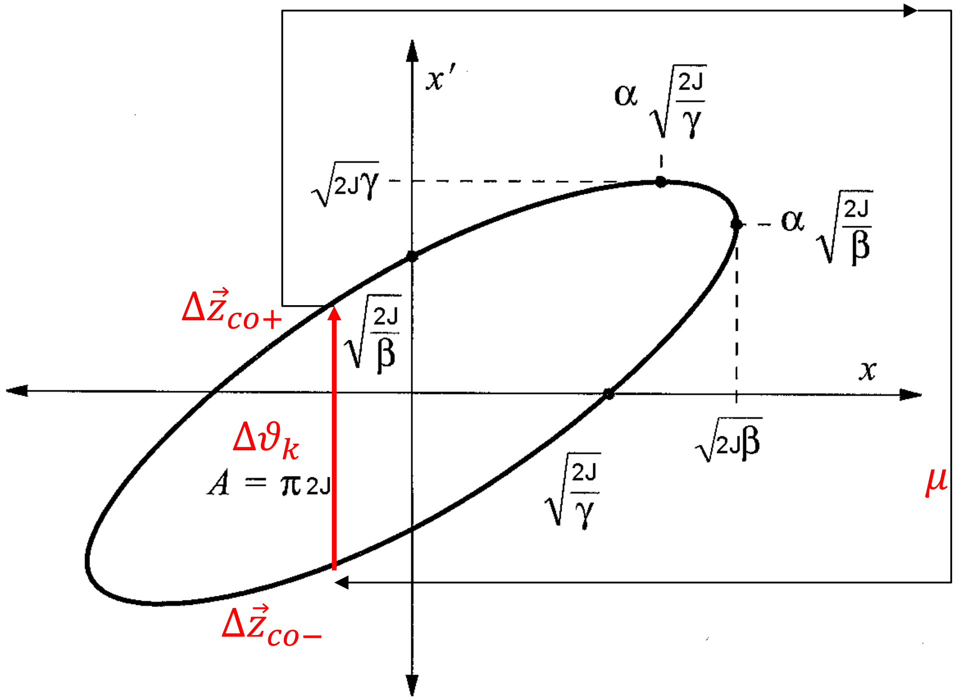
The oscillation amplitude J diverges when the tune ν is close to an integer.

$$s < s_k : \varphi_0 = \tfrac{\pi}{2} - \psi_k + \tfrac{\mu}{2} \quad , \quad s > s_k : \varphi_0 = \tfrac{\pi}{2} - \psi_k - \tfrac{\mu}{2}$$

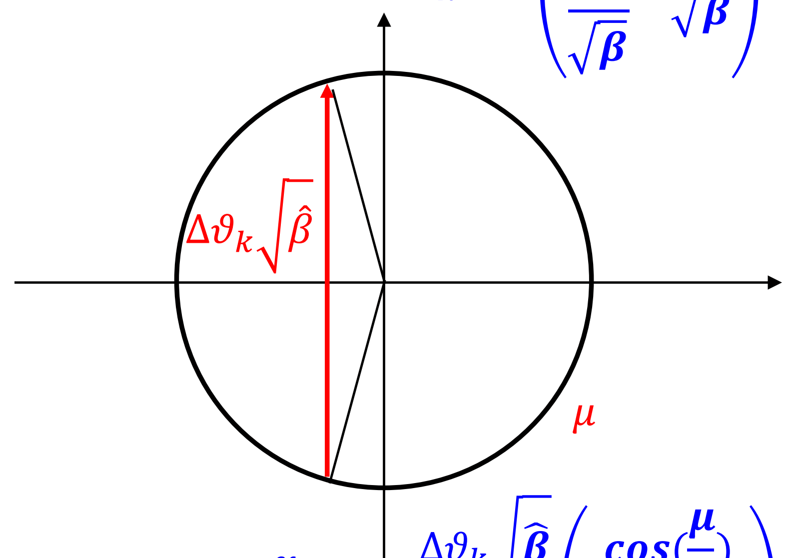
Phase jump by μ

Closed orbit for one kick

Free betatron oscillation



$$\begin{pmatrix}x_n\\ a_n\end{pmatrix} = \begin{pmatrix}\frac{1}{\sqrt{\beta}} & 0\\ \frac{\alpha}{\sqrt{\beta}} & \sqrt{\beta}\end{pmatrix}\begin{pmatrix}x\\ x'\end{pmatrix}$$



$$\begin{pmatrix}x_{n\pm}\\ a_{n\pm}\end{pmatrix} = \frac{\Delta\vartheta_k\sqrt{\hat{\beta}}}{\sin(\frac{\mu}{2})}\begin{pmatrix}\cos(\frac{\mu}{2})\\ \pm\sin(\frac{\mu}{2})\end{pmatrix}$$

$$x_n(s) = \frac{\Delta\vartheta_k\sqrt{\beta\hat{\beta}}}{2\sin(\frac{\mu}{2})}\cos(\frac{\mu}{2} - |\psi - \psi_k|)$$

$$\begin{pmatrix}x_{n\pm}\\ a_{n\pm}\end{pmatrix} = \frac{\Delta\vartheta_k\sqrt{\hat{\beta}}}{\sin(\frac{\mu}{2})}\begin{pmatrix}\frac{\mu}{2} - |\psi - \psi_k|\\ \dots\end{pmatrix}$$

Closed orbit correction in periodic accelerators

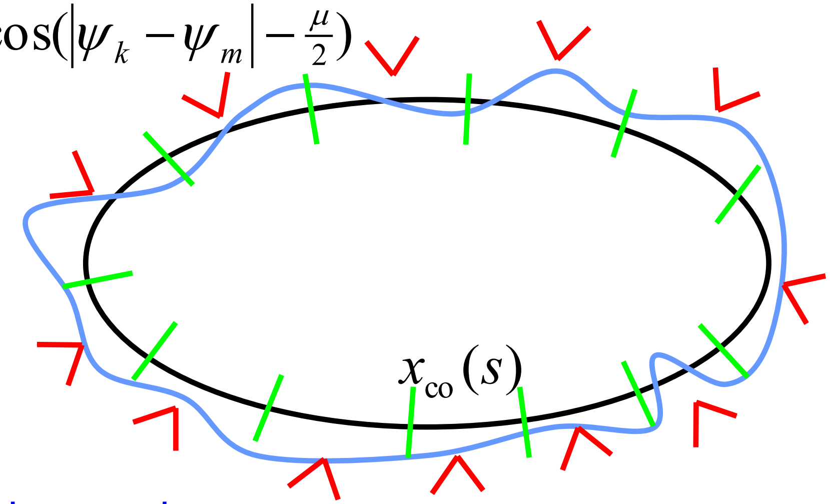
When the closed orbit $x_{co}^{old}(s_m)$ is measured at beam position monitors (BPMs, index m) and is influenced by corrector magnets (index k), then the monitor readings before and after changing the kick angles created in the correctors by $\Delta\vartheta_k$ are related by

$$x_{co}^{new}(s_m) = x_{co}^{old}(s_m) + \sum_k \Delta\vartheta_k \frac{\sqrt{\beta_m\beta_k}}{2\sin\frac{\mu}{2}}\cos(|\psi_k - \psi_m| - \tfrac{\mu}{2})$$

$$= x_{co}^{old}(s_m) + \sum_k O_{mk}\Delta\vartheta_k$$

$$\vec{x}_{co}^{new} = \vec{x}_{co}^{old} + \underline{O}\Delta\vec{\vartheta}$$

$$\Delta\vec{\vartheta} = -\underline{O}^{-1}\vec{x}_{co}^{old} \quad \Rightarrow \quad \vec{x}_{co}^{new} = 0$$

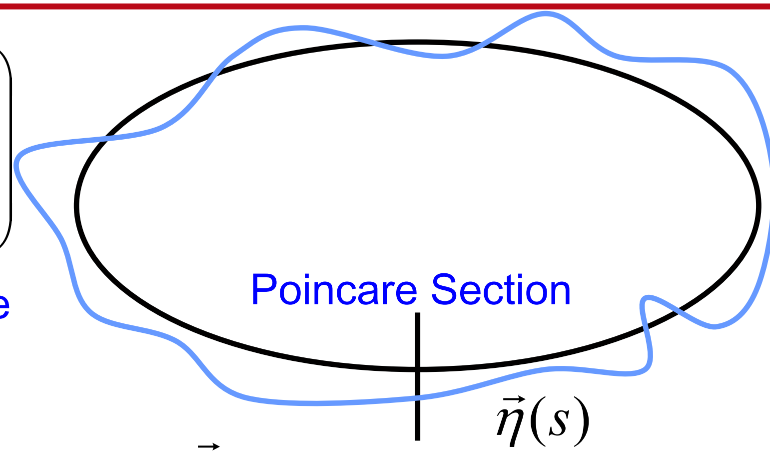


It is often better not to try to correct the closed orbit at the the BPMs to zero in this way since

1. computation of the inverse can be numerically unstable, so that small errors in the old closed orbit measurement lead to a large error in the corrector coil settings.
2. A zero orbit at all BPMs can be a bad orbit inbetween BPMs

Periodic dispersion

$$\begin{pmatrix} \underline{M}_{0x}\vec{z}_0 + \vec{D}(L)\delta \\ M_{56}\delta \\ \delta \end{pmatrix} = \begin{pmatrix} \underline{M}_{0x} & \vec{0} & \vec{D}(L) \\ \vec{T}^T & 1 & M_{56} \\ \vec{0}^T & 0 & 1 \end{pmatrix} \begin{pmatrix} \vec{z}_0 \\ 0 \\ \delta \end{pmatrix}$$



The periodic orbit for particles with relative energy deviation δ is

$$\vec{\eta}(0) = \underline{M}_0\vec{\eta}(0) + \vec{D}(L) \qquad \vec{\eta}(L) = \underline{M}_0\vec{\eta}(0) + \vec{D}(L) \quad \text{with} \quad \vec{\eta}(L) = \vec{\eta}(0)$$

$$\Downarrow$$

$$\boxed{\vec{\eta}(0) = [\underline{1} - \underline{M}_0(0)]^{-1}\vec{D}(L)}$$

Particles with energy deviation δ oscillates around this periodic orbit.

$$\vec{z} = \vec{z}_\beta + \delta\vec{\eta}$$

$$\vec{z}_\beta(L) + \delta\vec{\eta}(L) = \vec{z}(L) = \underline{M}_0\vec{z}(0) + \vec{D}(L)\delta = \underline{M}_0[\vec{z}_\beta(0) + \delta\vec{\eta}(0)] + \vec{D}(L)\delta$$

$$= \underline{M}_0\vec{z}_\beta(0) + \delta\vec{\eta}(L)$$

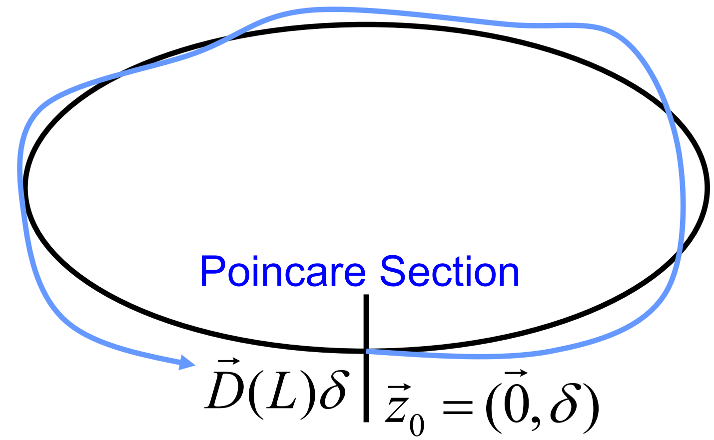
Periodic dispersion integral

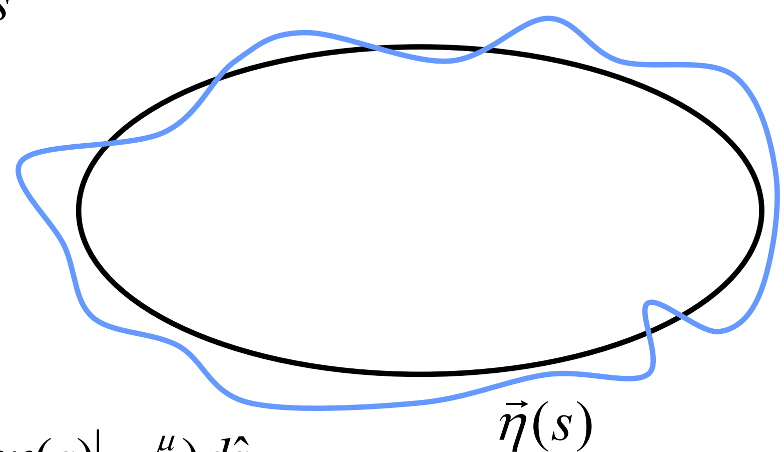
$$x' = a$$

$$a' = -(\kappa^2 + k)x + \kappa\delta$$

$$\vec{z} = \underline{M}\vec{z}_0 + \int_0^s \underline{M}(s-\hat{s})\begin{pmatrix} 0 \\ \delta\kappa(\hat{s}) \end{pmatrix} d\hat{s}$$

$$\Rightarrow \quad \vec{D}(L) = \int_0^L \underline{M}(L-\hat{s})\begin{pmatrix} 0 \\ \kappa(\hat{s}) \end{pmatrix} ds'$$





$$\Delta\kappa = \delta\kappa$$

$$\eta(s) = \frac{\sqrt{\beta(s)}}{2\sin\frac{\mu}{2}} \oint \kappa(\hat{s})\sqrt{\beta(\hat{s})}\cos(|\psi(\hat{s}) - \psi(s)| - \tfrac{\mu}{2})d\hat{s}$$

Variation of constants

$$\vec{z}\,' = \vec{f}(\vec{z}, s)$$

$$\vec{z}\,' = \underline{L}(s)\vec{z} + \Delta\vec{f}(\vec{z}, s)$$ Field errors, nonlinear fields, etc can lead to $\Delta\vec{f}(\vec{z}, s)$

$$\vec{z}\,'_H = \underline{L}(s)\vec{z}_H \quad \Rightarrow \quad \vec{z}_H(s) = \underline{M}(s)\vec{z}_{H0} \quad \text{with} \quad \underline{M}'(s)\vec{a} = \underline{L}(s)\underline{M}(s)\vec{a}$$

$$\vec{z}(s) = \underline{M}(s)\vec{a}(s) \quad \Rightarrow \quad \vec{z}\,'(s) = \underline{M}'(s)\vec{a} + \underline{M}(s)\vec{a}\,'(s) = \underline{L}(s)\vec{z} + \Delta\vec{f}(\vec{z}, s)$$

$$\vec{a}(s) = \vec{z}_0 + \int_0^s \underline{M}^{-1}(\hat{s})\Delta\vec{f}(\vec{z}(\hat{s}), \hat{s})\, d\hat{s}$$

$$\vec{z}(s) = \underline{M}(s)\left\{\vec{z}_0 + \int_0^s \underline{M}^{-1}(\hat{s})\Delta\vec{f}(\vec{z}(\hat{s}), \hat{s})\, d\hat{s}\right\}$$

$$= \vec{z}_H(s) + \int_0^s \underline{M}(s-\hat{s})\Delta\vec{f}(\vec{z}(\hat{s}), \hat{s})\, d\hat{s}$$

Perturbations are propagated from s to s'

Qadrupole errors in one-pass accelerators

$$\vec{z}' = \underline{L}(s)\vec{z} + \Delta\vec{f}(\vec{z}, s)$$

$$\vec{z}(s) = \vec{z}_H(s) + \int_0^s \underline{M}(s,\hat{s})\Delta\vec{f}(\vec{z},\hat{s})\,d\hat{s} \approx \vec{z}_H(s) + \int_0^s \underline{M}(s,\hat{s})\Delta\vec{f}(\vec{z}_H,\hat{s})\,d\hat{s}$$

$$x'' = -(\kappa^2 + k)x - \Delta k(s)\,x \quad \Rightarrow \quad \begin{pmatrix} x' \\ a' \end{pmatrix} = \begin{pmatrix} a \\ -(\kappa^2 + k)x \end{pmatrix} - \begin{pmatrix} 0 & 0 \\ \Delta k(s) & 0 \end{pmatrix}\begin{pmatrix} x \\ a \end{pmatrix}$$

$$\vec{z}(s) = \left\{ \underline{M}(s) - \int_0^s \underline{M}(s,\hat{s}) \begin{pmatrix} 0 & 0 \\ \Delta k(\hat{s}) & 0 \end{pmatrix} \underline{M}(\hat{s},0)\,d\hat{s} \right\} \vec{z}_0$$

One quadrupole error:

$$\Delta\underline{M}(s,\hat{s}) = -\underline{M}(s,\hat{s}) \begin{pmatrix} 0 & 0 \\ \Delta kl(\hat{s}) & 0 \end{pmatrix}$$

Qadrupole errors and Twiss in one pass accelerators

$$\Delta \underline{M}(s,\hat{s}) = -\underline{M}(s,\hat{s})\begin{pmatrix} 0 & 0 \\ \Delta kl(\hat{s}) & 0 \end{pmatrix} \qquad \underline{M}(s) = \begin{pmatrix} \dots & \sqrt{\beta_0\beta}\sin\tilde{\psi} \\ \dots & \sqrt{\frac{\beta_0}{\beta}}[\cos\tilde{\psi} - \alpha\sin\tilde{\psi}] \end{pmatrix}$$

$$\Delta \underline{M}(s,\hat{s}) = -\Delta kl(\hat{s})\begin{pmatrix} \sqrt{\hat{\beta}\beta}\sin\psi & 0 \\ \sqrt{\frac{\hat{\beta}}{\beta}}[\cos\psi - \alpha\sin\psi] & 0 \end{pmatrix}, \qquad \psi = \psi(s) - \psi(\hat{s})$$

$$= \begin{pmatrix} \frac{\frac{1}{2}\Delta\beta[\cos\psi + \hat{\alpha}\sin\psi] + \Delta\psi\beta[\hat{\alpha}\cos\psi - \sin\psi]}{\sqrt{\hat{\beta}\beta}} & \sqrt{\hat{\beta}}\left(\frac{\frac{\Delta\beta}{2}\sin\psi + \Delta\psi\beta\cos\psi}{\sqrt{\beta}}\right) \\ \dots & \dots \end{pmatrix}$$

$$\tfrac{1}{2}\Delta\beta\cos\psi + \tfrac{1}{2}\Delta\beta\frac{\sin^2\psi}{\cos\psi} = \tfrac{1}{2}\Delta\beta\frac{1}{\cos\psi} = -\Delta kl(\hat{s})\beta\hat{\beta}\sin\psi \qquad \Delta\psi = -\frac{\Delta\beta}{2\beta}\tan\psi$$

$$\boxed{\Delta\beta = -\Delta kl(\hat{s})\beta\hat{\beta}\sin 2\psi} \qquad \boxed{\Delta\psi = \Delta kl(\hat{s})\hat{\beta}\tfrac{1}{2}(1 - \cos 2\psi)}$$

Twiss changes in one-pass accelerators

$$\Delta\psi = \Delta k l_j \beta_j \sin^2(\psi - \psi_j)$$

→ More focusing always increases the tune

$$\frac{\Delta\beta}{\beta} = -\Delta k l_j \beta_j \sin(2[\psi - \psi_j])$$

→ Beta beat oscillates twice as fast as orbit.

Notice the self consistency: $\Delta\psi = \int_0^s \left(\frac{1}{\hat\beta + \Delta\hat\beta} - \frac{1}{\hat\beta}\right) d\hat s = -\int_0^s \frac{\Delta\hat\beta}{\hat\beta^2} d\hat s = -\int_0^s \frac{\Delta\hat\beta}{\hat\beta} d\hat\psi$

$$\Delta\alpha = -\frac{\Delta\beta'}{2} = \Delta k l_j \beta_j [\cos(2[\psi - \psi_j]) - \alpha \sin(2[\psi - \psi_j])]$$

$$\begin{pmatrix} \frac{\Delta\beta}{\beta} \\ \frac{\beta\Delta\alpha - \alpha\Delta\beta}{\beta} \end{pmatrix} = \Delta k l_j \, \beta_j \begin{pmatrix} \sin(2[\psi - \psi_j]) \\ \cos(2[\psi - \psi_j]) \end{pmatrix}$$

$$\left(\Delta k l_j \beta_j\right)^2 = [\left(\frac{\Delta\beta}{\beta}\right)^2 + \left(\Delta\alpha - \alpha\frac{\Delta\beta}{\beta}\right)^2]$$

Twiss correction in one-pass accelerators

$$\frac{\Delta\beta}{\beta} = -\sum_j \Delta kl_j \beta_j \sin(2[\psi - \psi_j]) \qquad \Delta\psi = \sum_j \Delta kl_j \beta_j \tfrac{1}{2}[1 - \cos(2[\psi - \psi_j])]$$

When beta functions and betatron phases have been measured at many places, quadrupoles can be changed with these formulas to correct the Twiss errors.

Quadruple errors and the tune

$$\cos(\mu+\Delta\mu)=\tfrac{1}{2}\mathrm{Tr}[M_0(s_j)+\Delta M_0(s_j)]\approx\cos\mu-\Delta\mu\sin\mu$$

$$=\tfrac{1}{2}\mathrm{Tr}[\begin{pmatrix}1 & 0\\ -\Delta kl_j & 1\end{pmatrix}\begin{pmatrix}\cos\mu+\alpha_j\sin\mu & \beta_j\sin\mu\\ -\gamma_j\sin\mu & \cos\mu-\alpha_j\sin\mu\end{pmatrix}]$$

$$=\cos\mu-\tfrac{1}{2}\Delta kl_j\beta_j\sin\mu$$

$$\boxed{\Delta\mu=\tfrac{1}{2}\Delta kl_j\beta_j}$$

Oscillation frequencies can be measured relatively easily and accurately.

Measurement of beta function: Change k and measure tune.

Quadruple errors and periodic beta function

One pass accelerators:

$$\Delta x(s) = \Delta\vartheta\sqrt{\beta\hat{\beta}}\sin(\psi - \hat{\psi})$$

$$\frac{\Delta\beta}{\beta} = -\Delta k l \hat{\beta}\sin(2(\psi - \hat{\psi}))$$

Periodic accelerators:

$$\Delta x_{co}(s) = \frac{\Delta\vartheta\sqrt{\beta\hat{\beta}}}{\sin(\frac{\mu}{2})}\cos(|\psi - \hat{\psi} - \frac{\mu}{2})$$

$$\frac{\Delta\beta}{\beta} = -\frac{\Delta k l \hat{\beta}}{2\sin(\mu)}\cos(2|\psi - \hat{\psi}| - \mu)$$

Energy dependent Twiss parameters

Natural Chromaticity: $\xi_x = \frac{1}{2\pi}\partial\mu_x/\partial\delta\ ,\ \xi_y = \frac{1}{2\pi}\partial\mu_y/\partial\delta$

$$\Delta\mu_x = \frac{1}{2}\Delta k l \hat{\beta}_x \longrightarrow \partial\mu_x/\partial\delta = -\frac{1}{2}\int_0^L k(s)\beta_x(s)\,ds$$

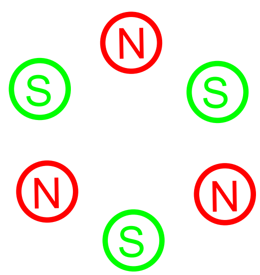
$$\Delta\mu_y = \frac{1}{2}\Delta k l \hat{\beta}_y \longrightarrow \partial\mu_y/\partial\delta = \frac{1}{2}\int_0^L k(s)\beta_y(s)\,ds$$

The periodic beta functions will similarly depend on energy.

Sextupoles (revisited)

$$\psi = \Psi_3 \,\mathrm{Im}\{(x-iy)^3\} = \Psi_3 \cdot (y^3 - 3x^2 y) \quad \Rightarrow \quad \vec{B} = -\vec{\nabla}\psi = \Psi_3\, 3 \begin{pmatrix} 2xy \\ x^2 - y^2 \end{pmatrix}$$

C_3 Symmetry



i) Sextupole fields hardly influence the particles close to the center, where one can linearize in x and y.

ii) In linear approximation a by Δx shifted sextupole has a quadrupole field.

iii) When Δx depends on the energy, one can build an energy dependent quadrupole.

$$\vec{B} = -\vec{\nabla}\psi = \Psi_3\, 3 \begin{pmatrix} 2xy \\ x^2 - y^2 \end{pmatrix}$$

$$x \mapsto \Delta x + x$$

$$\vec{B} \approx \Psi_3\, 3 \begin{pmatrix} 2xy \\ x^2 - y^2 \end{pmatrix} + 6\Psi_3 \Delta x \begin{pmatrix} y \\ x \end{pmatrix} + O(\Delta x^2)$$

$$k_2 = \frac{q}{p} 3!\Psi_3 \Rightarrow k_1 = k_2 \Delta x$$

Chromaticity and its correction

Chromaticity ξ = energy dependence of the tune

$$\upsilon(\delta) = \upsilon + \tfrac{\partial\upsilon}{\partial\delta}\delta + \dots$$

$$\xi = \tfrac{\partial\upsilon}{\partial\delta} \quad \text{with} \quad \upsilon = \tfrac{\mu}{2\pi}$$

Natural chromaticity ξ_0 = energy dependence of the tune due to quadrupoles only

$$\xi_{x0} = -\frac{1}{4\pi}\oint \beta_x(\hat{s})k_1(\hat{s})d\hat{s} \qquad \xi_{y0} = \frac{1}{4\pi}\oint \beta_y(\hat{s})k_1(\hat{s})d\hat{s}$$

Particles with energy difference oscillate around the periodic dispersion leading to a quadrupole effect in sextupoles that also shifts the tune:

$$\xi_x = \frac{1}{4\pi}\oint \beta_x(-k_1 + \eta_x k_2)d\hat{s} \qquad \xi_y = \frac{1}{4\pi}\oint \beta_y(k_1 - \eta_x k_2)d\hat{s}$$

Typically the the chormaticity ξ is chosen to be slightly positive, between 0 and 3.

Chromatic beta beat minimization

Chromatic beta beat is mostly created by the strongest quadrupoles, it can be influenced by sextupoles, under the provision that these on average still correct the chromaticity.

Linear accelerators:

$$\frac{d\beta}{d\delta} = \beta(k_1 l - k_2 l \cdot \eta)\hat{\beta}\sin(2|\psi - \hat{\psi}| - \mu)$$

Periodic accelerators:

$$\frac{d\beta}{d\delta} = \frac{\beta}{2\sin\mu}(k_1 l - k_2 l \cdot \eta)\hat{\beta}\cos(2|\psi - \hat{\psi}| - \mu)$$

Sextupoles are used to compensate the chromaticity.

Several sextupoles are used to have their average compensate the chromaticity but have their regional variation compensate the chromatic beta beat on average and at critical sections, e.g. interaction points.