This tutorial will cover the following topics

1. Collider & FEL applications criteria
2. Review of fundamental coordinate system, normalized units & constant of motion
3. Physics of beam loading, Low energy spread, & efficiency
4. Physics of particle trapping

→ These notes are based USPAS 2022 lecture notes taught by myself, Alec Thomas (university of Michigan) & Warren Mori (UCLA). website:

https://sites.google.com/umich.edu/uspas-plasma-accelerators/home

Luminosity

- In addition to energy, beam quality is a critical consideration
- In colliders, quality is measured by Luminosity

Beam collision rate ⟹ High rate

PWFA Collider: f~30 kHz

Beam charge ⟹ High charge

PWFA Collider: Q~1.6 nC , 10^{10} e^-

$$L \sim \frac{fN^2}{4\pi\sigma^2}$$

Focusability ⟹ low emittance and energy spread

PWFA Collider:

- Emittance, ϵ_n~ 35 nm
- Energy spread, $\Delta E/E < 1\%$

$L\uparrow \Longrightarrow \epsilon_n \downarrow$ & $\Delta E/E \downarrow$ & Q↑

http://lhc-closer.es/taking_a_closer_look_at_lhc/0.luminosity/idioma/en_GB

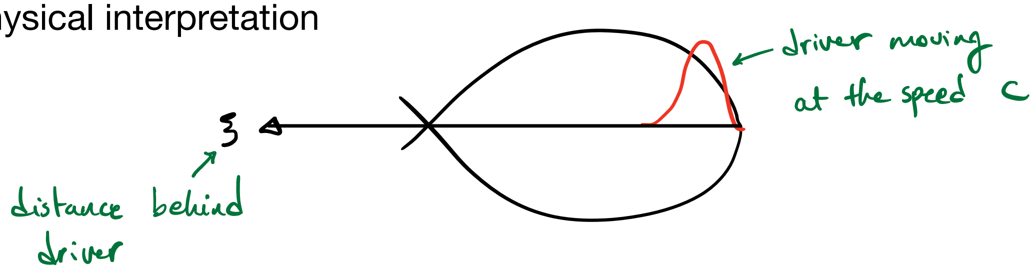
J.-P. Delahaye, Proc. IPAC 2013, TUPME020

The co-moving coordinates

This is one of the most important concepts and a source of much confusion for those who are just starting out the study in this field. Because this fields includes drivers of physical phenomena that are moving at near the speed of light, many variables depend on the quantity $ct - z$, rather than on 't' or 'z' alone. Therefore, a new coordinate system is developed to work with this explicit dependence:

$$\text{cartesian: } \begin{array}{c} x \\ y \\ z \\ t \end{array} \Longrightarrow \text{co-moving frame: } \begin{array}{l} x' = x \\ y' = y \\ t' = t \\ \xi = ct - z \end{array}$$

Physical interpretation



a particle moving at the speed of light will maintain its position in ξ while it is moving in the Cartesian coordinates.

Note that while you may see this coordinate transform being referred to as "going to the speed of light frame", there are no Lorentz transforms performed in this operation, and as such, this is **not** a proper change of frame. I prefer to call this "a change of coordinate systems to a co-moving variable", because this operation is simply a relabeling of variable that allows for much more intuitive interpretation of the equations of motion and fields.

Derivatives in this new coordinate system can be obtained with the use of chain rule:

$$\frac{\partial}{\partial t} = \frac{\partial}{\partial t'} + c\frac{\partial}{\partial \xi}$$

$$\frac{\partial}{\partial z} = -\frac{\partial}{\partial \xi}$$

Total time derivative: this is meaningful when all the variables are a function of time, which is the case when we look at a single particle. In that case,

Total ξ derivative: Since $\xi = ct - z$, where $z \equiv z(t)$, e.g. where we follow a particle, total ξ derivative for a function of ξ can be expressed as:

$$\frac{d f(\xi)}{dt} = \frac{df}{d\xi}\frac{d\xi}{dt} = (c - v_z)\frac{df}{d\xi}$$

Note that from above definitions,

$$\frac{\partial}{\partial t} + c\frac{\partial}{\partial z} \longrightarrow \frac{\partial}{\partial t'}$$

This means that the wave equation operator becomes

$$\frac{\partial^2}{\partial t^2} - c^2\nabla^2 = \frac{\partial^2}{\partial t^2} - c^2\frac{\partial^2}{\partial z^2} - c^2\underbrace{\nabla_\perp^2}_{x \,\&\, y \text{ components}}$$

$$= \frac{\partial^2}{\partial t'^2} + 2c\frac{\partial}{\partial t'}\frac{\partial}{\partial \xi} - c^2\nabla_\perp^2$$

Quasi-static approximation: a common assumption in wakefield theory, it states that physical quantities in co-moving coordinates change very slowly in time.

Mathematically, $\frac{\partial}{\partial t'} \ll \frac{\partial}{\partial \xi}$,

$$\frac{\partial}{\partial t} \rightarrow \xi\frac{\partial}{\partial \xi}$$

$$\frac{\partial^2}{\partial t^2} - c^2\nabla^2 \longrightarrow -c^2\nabla_\perp^2$$

the wave equation reduces to a 2D Poisson solution

Normalizations:

In this section, we made extensive use of the normalized parameters, where we divided a quantity by a "natural scale", e.g. P/mc. Normalized quantities are very useful because they allow you to understand the relative strength of a parameter, e.g. a laser pulse with $a_0 \sim 0.01$ creates only a minor perturbation (regardless of what combination of intensity and wavelength create that value of a_0)whereas a laser with $a_0 \sim 10$ creates highly nonlinear phenomena. The normalized parameters are used extensively in description of plasma accelerators, so here we review them for the quantities in our studies.

We have already seen the natural scales for velocity and momentum:

$$P \longrightarrow \frac{P}{mc} \qquad\qquad v \longrightarrow \frac{v}{c}$$

For time, we need a normalizing frequency:

$$t \rightarrow \omega_N t \rightarrow \omega_p t$$

This also sets the space normalization:

$$\vec{x} \rightarrow \vec{x}\frac{\omega_N}{c} \rightarrow \vec{x}\frac{\omega_p}{c}$$

To get the normalizations for fields & charge density, let's normalize the equation of motion:

$$\frac{1}{mc}\frac{d\vec{p}}{dt\cdot\omega_N} = \frac{qE}{mc\omega_N} + \frac{\vec{v}}{c}\times\frac{q\vec{B}}{m\omega_N}$$

The normalized quantities are:

$$\frac{e\vec{E}}{mc\omega_N} \longrightarrow \vec{E}$$

$$\frac{e\vec{B}}{m\omega_N} \longrightarrow \vec{B}$$

Recall $\omega_c = \frac{e\vec{B}}{m}$ is the cyclotron frequency of e^- in field $\vec{B}$.

Particle number density is commonly used as part of ρ & $\vec{J}$. In the context of plasma, we normalize this variable to the background (or initial) density:

$$\frac{n}{n_0} \longrightarrow n$$

What about permittivity/permeability of free space?

$$\frac{c}{\omega_N}\cdot\vec{\nabla}\cdot\frac{e\vec{E}}{mc\omega_N} = \frac{\rho}{\epsilon_0} = \frac{q}{e}\frac{e^2}{m}\frac{n_0}{\epsilon_0}\frac{n}{n_0}\cdot\frac{1}{\omega_N^2}$$

1/spatial units

Since $\omega_p^2 = \frac{e^2 n_0}{m\epsilon_0}$, $\frac{1}{\epsilon_0}$ is normalized as follows:

$$\frac{e^2 n_0}{m\epsilon_0}\cdot\frac{1}{\omega_N^2} = \frac{\omega_p^2}{\omega_N^2} \rightarrow 1$$

Using Ampere's law, it can be shown that once normalized, μ_0 will also become

$$\mu_0 \rightarrow \frac{\omega_p^2}{\omega_N^2} \rightarrow 1$$

For the scalar and vector potential, definitions can be derived from the definition of the electric field,

$$\frac{e\vec{A}}{mc} \rightarrow \vec{A}$$

$$\frac{e\phi}{mc^2} \rightarrow \phi$$

wake function defined as $\psi = \phi - cA_z$ is a very important function & is normalized as:

$$\frac{e\psi}{mc^2} \rightarrow \psi,$$ sometimes written as upper case Φ

<u>Electrodynamics in the co-moving coordinates</u>

We start from the equation for potentials in the Lorentz Gauge:

$$\begin{cases} \nabla^2\phi - \frac{1}{c^2}\frac{\partial^2\phi}{\partial t^2} = -\frac{\rho}{\epsilon_0} \\ \nabla^2\vec{A} - \frac{1}{c^2}\frac{\partial^2\vec{A}}{\partial t^2} = -\mu_0\vec{J} \end{cases}$$

Gauge condition $\Rightarrow$ $\vec{\nabla}\cdot\vec{A} + \frac{1}{c}\frac{\partial\phi}{\partial t} = 0$

in the co-moving coordinate under the quasistatic approximation

$\left(\frac{\partial}{\partial t'} \to 0\right)$, wave operator $\to -\nabla_\perp^2$

$$\begin{cases} \nabla_\perp^2\phi = -\frac{\rho}{\epsilon_0} \\ \nabla_\perp^2\vec{A} = -\mu_0\vec{J} \end{cases}$$

in normalized units $\Rightarrow$

$$\begin{cases} \nabla_\perp^2\phi = -\rho & (1) \\ \nabla_\perp^2\vec{A} = -\vec{J} & (2) \end{cases}$$

Gauge condition $\Rightarrow$ $\vec{\nabla}_\perp\cdot\vec{A}_\perp + \underbrace{\frac{\partial A_z}{\partial z}}_{= -\partial_\xi A_z} + \underbrace{\frac{1}{c}\frac{\partial\phi}{\partial t}}_{= \partial_\xi\phi} = 0$ $\quad(\partial_{t}' = 0)$

$$\Rightarrow \partial_\xi\psi + \vec{\nabla}_\perp\cdot\vec{A}_\perp = 0 \quad \ldots (3)$$

where $\psi = \phi - A_z$

Eqn 1 and 2 are used to get the Poisson equation for wake function, ψ in the do-moving coordinates

$$\boxed{\nabla_\perp^2(\phi - A_z) = \nabla_\perp^2\psi = -(\rho - J_z)} \quad (4)$$

Finally, the source terms are connected by the continuity equation:

$$\frac{\partial\rho}{\partial t} + \vec{\nabla}\cdot\vec{J} = 0$$

In terms of co-moving coordinates & in normalized units,

$$\frac{\partial}{\partial\xi}(\rho - J_z) + \vec{\nabla}_\perp\cdot\vec{J}_\perp = 0 \quad (5)$$

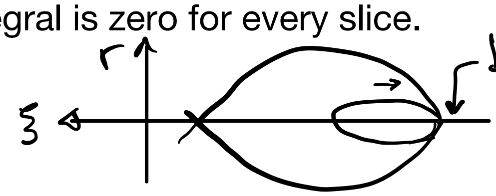
Because of the 2D Poisson equations, we can imagine that what happens in each slice as independent from the other slices. Eqn 5 is the continuity equation in that world. So if we integrate over all space in this equation (i.e. in 2D), the second term drops out from Gauss's law evaluated over the boundary of infinite space, leaving the first term as a new conserved quantity, I.e.

$$\int \frac{\partial}{\partial\xi}(\rho - J_z)\, dx_\perp + \underbrace{\int \vec{\nabla}_\perp\cdot\vec{J}_\perp\, dx_\perp}_{\oint_{\infty\text{ boundary}} \vec{J}_\perp\cdot d\vec{l}_\perp = 0} = 0$$

$$\Rightarrow \boxed{\frac{d}{d\xi}\int(\rho - J_z)\, dx_\perp = 0}$$

The integral over all space in $x_\perp$ can depend only on ξ

This is analogous to total charge in a regular problem, which is conserved. In this problem now, it is the $\int \rho - J_z\, dx_\perp$ that is conserved from one slice to the next. Recall that ξ , the co-moving coordinate is the distance along the driver. Since charge and current density are zero for the slice immediately ahead of the driver, this integral is zero for every slice.



The continuity equation in the co-moving coordinate then becomes

$$\boxed{\int(\rho - J_z)\, dr = 0} \text{ for each } \xi \text{ slice} \quad \cdots (6)$$

So now, to solve our 2D Poisson equations, we could use our electrostatic intuition. Suppose we have the following situation

$-\nabla_\perp^2 f = S$ w/ boundary condition that $f \to 0$ as $r \to \infty$, which is the case for ϕ, A, & ψ

Using insights from the solution of azimuthally symmetric ∞ cylinder & the resulting electrostatic potential, we can write

$$f = f_0(\xi) + f_1(r)$$

$$\text{where } \begin{cases} f_0 = \int_0^\infty dr' \frac{1}{r'} \int_0^{r'} dr'' \, r'' \, S(r'') \\ f_1 = \int_0^r [\qquad\qquad] \end{cases} \quad \ldots (7)$$

i.e. we can decompose each potential into a ξ-dependent & a r-dependent component. $\Rightarrow$ $\psi = \psi_0(\xi) + \psi_1(r)$

Fields: $\vec{E} = -\vec{\nabla}\phi - \frac{\partial \vec{A}}{\partial t}$ co-moving coordinate

$\left(\frac{\partial}{\partial t'} = 0\right) \Rightarrow E_z = \frac{\partial \phi}{\partial \xi} - \frac{\partial A_z}{\partial \xi}$

$= \frac{\partial}{\partial \xi}(\phi - A_z)$

$$\boxed{E_z = \partial_\xi \psi = \frac{d\psi_0}{d\xi}} \quad (8)$$

$\vec{B} = \vec{\nabla} \times \vec{A}$

$B_r = \frac{1}{r}\left(\partial_\phi A_z - r\,\partial_z A_\phi\right) = 0$

$B_z = \frac{1}{r}\left(\partial_r r A_\phi - r\,\partial_\phi A_r\right) = 0$

Since $\frac{\partial}{\partial \phi} = 0$ & $\partial_\phi = 0$

For a relativistic electron accelerated by the wakefield, $\vec{v} \approx v_z \hat{z} \approx c\,\hat{z}$ the longitudinal force is simply $F_z = q E_z$. According to Eqn 8 then, the goal of low energy spread can be expressed as creating conditions for a constant slope for ψ_0 as a function of ξ. These conclusions are independent of the regime (linear/nonlinear, etc) and are independent of the type of the driver, laser, e-beam, etc.

<u>Conservation laws:</u>

There are several ways to reach the following equations, most simply perhaps by expressing the equation of motion in terms of the derivatives of the potential. The two conservation equations commonly used are the conservation of canonical momentum:

$$\frac{d}{dt}(\vec{p} + q\vec{A}) = -\vec{\nabla}\phi - q(\vec{\nabla}\vec{A}).\vec{v}$$

If the gradients of the problem are zero in any direction, the canonical momentum is conserved. This is often very useful for laser-driven wakefield, where under the assumption of a wide beam, $\nabla_\perp \approx 0$, momentum of the particle can be directly connected to the value of the vector potential.

Second conservation law is the conservation of the Hamiltonian of the particles, which is normalized units is written as

$$\frac{d}{dt}\mathcal{H} = \frac{d}{dt}\left[\gamma - p_z + q\psi\right] = 0 \quad \ldots (9)$$

Beam Loading

Loading particles into the wakefield absorbs energy from the wake. The absorbed energy is manifest in the reduction of the electric field available for accelerating the electrons. This effect is called beam loading. We have three objectives in beam loading:

1. Load as much charge as possible to increase the accelerated beam brightness
2. Minimize the energy spread of the accelerated beam. Since we are intending to inject substantial amount of charge into the wakefield, these charges are going to have a finite size both longitudinally and transversely. Therefore we want these electrons to be loaded in a way that the accelerating field felt by the entire beam is the same. Doing so will ensure a small final energy spread for the beam.
3. Maximize efficiency: the requirement of high luminosity also requires high energy extraction efficiency from the plasma by the beam

We will treat the two cases of linear and nonlinear beam loading separately.

We start from the discussion of the linear theory. The impact of loading a trailing particle beam in the plasma wakefield in linear theory can be calculated using the superposition principle as was shown by Katsouleas et, al in 1987; i.e. the wakefield due to the driver and trailing beam is separately calculated and the total wakefield is the sum of the fields of the individual beams.

Using the equations of linear fluid theory, a differential equation can be derived for the wake function in 1D

$$(\partial_\xi^2 + 1)\,\psi = \rho_b$$

where ρ_b is the charge density of the injected charge

The solution is given in terms of a Green's function

$$G(\xi) = \underbrace{\eta}_{\text{Heaviside step function}}(\xi - \xi')\,\sin(\xi - \xi')$$

$$\therefore\ \psi = \int_{-\infty}^{\infty} d\xi'\,\eta(\xi-\xi')\,\sin(\xi-\xi')\,\rho(\xi')$$

$$\therefore\ E = \frac{\partial\psi}{\partial\xi} = \int_{-\infty}^{\infty} d\xi'\,\eta(\xi-\xi')\,\cos(\xi-\xi')\,\rho(\xi')$$

a second term will include $\delta(\xi-\xi')$ & is zero

The wake function behind a driver in the 1D linear regime has a sinusoidal form. For laser, the solution is the same except the charge density is replaced with

$$\phi_p = \frac{|a^2|}{4}, \quad \text{ponderomotive potential}$$

where the vector potential of the laser is assumed to have the form $\vec{A} = \mathrm{Re}\{a(\xi)\,e^{-ik_0\xi}\}$, with $a(\xi)$ representing the slowly varying laser amplitude.

The superposition of an initial wake by a driver and the wake of the beam load is given by

$$E_t = \underbrace{E_0\cos\xi}_{\text{intial wake (eqn 4)}} - \underbrace{\int_{\xi_0}^{\xi} d\xi'\,\rho(\xi')\cos(\xi-\xi')}_{\text{wake generated by beam}}$$

where ξ_0 is the front of the beam & E_t is the total field of the wake & the beam & is constant.

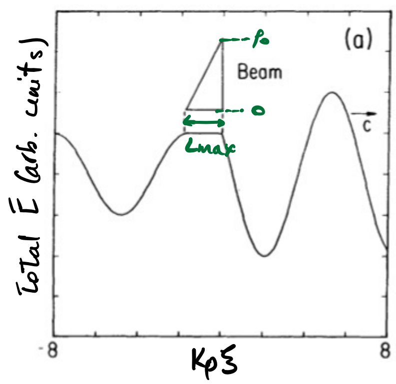
It turns out that the appropriate shape for obtaining a flat field is a ramped beam.

Assuming that the bunch density $\rho_b(\xi)$ has the form $a\xi + b$, begining from $\xi = \xi_0$ at the wave's minimum, and solving for a & b, we get

$$\rho_b(\xi) = -E_0\left[(\cos\xi_0)\,\xi + (\sin\xi_0 - \xi_0\cos\xi_0)\right]$$

Recall that in normalized units, $E_0 = n_1$, the density perturbation.

The results were shown in Fig 4(a) of the 1987 Katsoleaus paper reproduced here.



Katsouleas, 1987

The maximum charge is obtained by letting ρ_b go to zero (we restrict ρ_b to have only one sign)

The beam's properties in normalized units are:

peak beam density: $\rho_b(\xi_0) = \rho_b^{max} = -n_1 \sin\xi_0$

Max bunch length: $l_{max} = \tan\xi_0$

Accelerating field: $E_t = E_0 \cos(\xi_0)$

# of particles: $N = n_1 A \dfrac{\sin^2\xi_0}{2\cos\xi_0}$

density perturbation in normalized unit

cross sectional area of beam

Some limitations:

- The amount of charge is proportional to the cross sectional area of the beam. Because the fields of the wake are sinusoidal, the accelerating beam has to be narrow for emittance preservation. This reduces the charge and energy absorption efficiency
- Efficiency is proportional to the beam-loaded field as it is represented by the amplitude of the field ahead of the wake and after the wake. Therefore the higher the accelerated field, the lower the efficiency and also the lower the charge.

Energy absorbed per unit length:

$$E_t Q_b = \mathcal{E}_0 \cos(\xi_0)\, N_0 \frac{\sin^2 \xi_0}{2\cos \xi_0}$$

$$N_0 = n_1 A = \mathcal{E}_0 A$$

$$\sin^2 \xi_0 + \cos^2 \xi_0 = 1$$

$$\Rightarrow E_t Q_b = \underbrace{\frac{\mathcal{E}_0^2}{2}}_{} A - \underbrace{\frac{\mathcal{E}_t^2}{2}}_{} A$$

Def'n $\mathcal{E}_0$ (energy per unit length in front)

Def'n $\mathcal{E}_t$ (energy per unit length behind the bunch)

$$\eta_L = 1 - \frac{\mathcal{E}_t}{\mathcal{E}_0} = \begin{cases} 100\% & \text{for } \mathcal{E}_t = 0 \text{ (zero accelerating field)} \\ 0\% & \text{for } \mathcal{E}_t = \mathcal{E}_0 \text{ (max field)} \end{cases}$$

Let $A_{eff} \sim c^2/\omega_p^2$ to get an estimate for charge

$$\frac{1}{k_p} \equiv c/\omega_p \simeq 5.3\times 10^5 \Big/ \sqrt{n_0[\text{cm}^{-3}]}$$

Max particles: $N_0 \simeq \left(\frac{n_1}{n_0}\right) n_0 k_p^{-3} \simeq \frac{1.5\times 10^8}{\sqrt{n_0[10^{18}\,\text{cm}^{-3}]}} \left(\frac{n_1}{n_0}\right)$ fractional density perturbation

$\frac{n_1}{n_0} \ll 1$ – Linear theory assumption

Even setting $\frac{n_1}{n_0} = \frac{\delta n}{n} \sim 1$ to get an upper limit for N_0, we get

$$N_0 \sim 80\ \text{pC} \quad \text{for } n_0 = 10^{17}\,\text{cm}^{-3}$$

$$N_0 \sim 240\ \text{pC} \quad \text{for } n_0 = 10^{16}\,\text{cm}^{-3}$$

This is still clearly an overestimate, because we are assuming $\frac{\delta n}{n} \to 1$ & $A \to (c/\omega_p)^2$

In order to increase the loaded charge, we need to increase $\delta n/n$ & A_{eff}
To non-linear territory. We will next discuss beam loading and efficiency in the the 3D nonlinear wakes.

Nonlinear wakefield, blowout regime

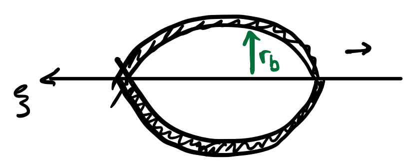
The other limit of interest is a highly nonlinear wakefield. In this case, the wake function, ψ, cannot be derived using Green's function. Instead, we specify the source term and solve for it using the 2D Poisson equation in the co-moving coordinate, i.e. equations 4,6, and 7

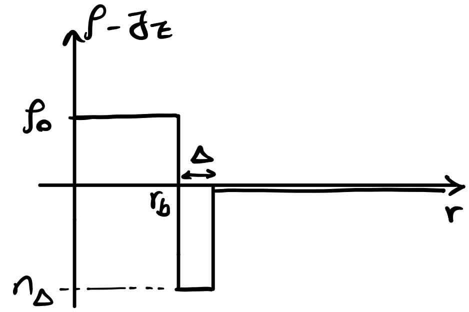
$$(4) \rightarrow \quad \nabla_{\perp}^2 \psi = -(\rho - J_z)$$

$$(6) \rightarrow \quad \int_0^{\infty} (\rho - J_z)\, dx_{\perp} = 0$$

$$(7) \rightarrow \begin{cases} \psi_0 = \displaystyle\int_0^{\infty} dr' \frac{1}{r'} \int_0^{r'} dr'' \, r'' \, (\rho - J_z) \\ \psi_1 = \displaystyle\int_0^{r} \Big[\qquad \Big] \end{cases}$$

From simulations, we know that in a highly nonlinear wakefield, each ξ slice is composed of an ion column out to a radius r_b surrounded by a sheath of e^-. Assuming ions are immobile, the $\rho - J_z$ can be constructed as follows





$r < r_b$, $\rho - J_z = \rho_0 = 1$ in normalized units
(Inside ion column)

$r \to \infty$, $\rho - J_z = 0$ (undisturbed plasma)

Using the continuity eqn,

$$\int_0^{\infty} (\rho - J_z)\, 2\pi r\, dr = 0 \implies n_\Delta = \frac{r_b^2}{(r_b + \Delta)^2 - r_b^2}$$

Using eqn 7, the wake function can be derived:

$$\begin{cases} \psi = \underbrace{(1+\beta)\frac{r_b^2}{4}}_{\psi_0} - \frac{r^2}{4} & \cdots (10) \\ \beta = \frac{(1+\alpha)^2 \ln(1+\alpha)^2}{(1+\alpha)^2 - 1} - 1 \\ \alpha = \frac{\Delta}{r_b} \end{cases}$$

For easier calculations, we consider the regime of $r_b \gg 1$, which corresponds to α & $\beta \to 0$ &

$$\boxed{\begin{aligned} \psi &= \psi_0(\xi) - \frac{r^2}{4} \\ &\approx \frac{r_b^2}{4} - \frac{r^2}{4} \end{aligned}} \quad \cdots (11)$$

This means that $E_z = \frac{\partial \psi}{\partial \xi} = \frac{d\psi_0}{d\xi} \approx \frac{1}{2} r_b \frac{dr_b}{d\xi}$

$$\boxed{E_z = \frac{1}{2} r_b r_b'} \quad \cdots (12)$$

The sheath radius, which can be considered as the transverse trajectory of the innermost electron can be derived using the transverse equation of motion, where all variables are written in terms of the wake potential, ψ. In the ultrarelativistic limit, where the sheath radius is very large, the result almost describes the equation of a circle:

$$r_b \frac{d^2 r_b}{d\xi^2} + 2\left(\frac{dr_b}{d\xi}\right)^2 + 1 = 0 \quad \cdots (13)$$

Comparing with the equation for a circle,

$$x^2 + y^2 = 1 \Rightarrow y\frac{d^2y}{dx^2} + \left(\frac{dy}{dx}\right)^2 + 1 = 0$$

In the presence of accelerating e^-, eqn 13 is modified to

$$\boxed{r_b \frac{d^2 r_b}{d\xi^2} + 2\left(\frac{dr_b}{d\xi}\right)^2 + 1 = \frac{4\lambda(\xi)}{r_b^2}} \cdots (14)$$

Lu, 2006
Tzoufras, 2009

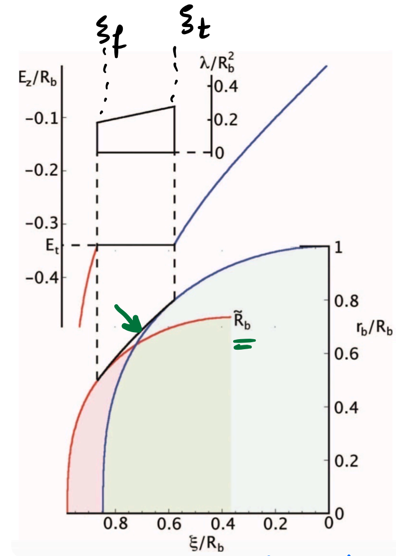
$$\lambda(\xi) = \int_0^\infty q\, r\, n_b(r)\, dr$$

(normalized) charge per unit length

First we discuss the physical picture of wake flattening this is accomplished.

Consider a trailing beam placed inside a wakefield with initial maximum bubble radius of R_b. To avoid nonlinear focusing fields, we require that this beam fit inside the bubble.

In presence of the trailing bunch, the trajectory is modified to correspond to the black curve in the region of the beam, which is determined by the requirement of constant field.



blue: absence of e^- bunch

The sheath after trailing beam feel only the ion column, but because energy was absorbed, the trajectory in that segment corresponds to a bubble with smaller maximum radius:

Analytical solutions:

We can express λ as a function of r_b instead of ξ

$$\Lambda(r_b) = \lambda[\xi(r_b)]$$

Rewrite (14) as

$$r_b'' = \frac{4\Lambda(r_b) - r_b^2\left[2(r_b')^2 + 1\right]}{r_b^3}$$

substituting $r'' = r_b'\left(\frac{dr_b'}{dr_b}\right)$ & integrating (see Appendix of Tzoufras 2009), we obtain

$$E_z = \frac{1}{2} r_b r_b' = -\frac{r_b}{2\sqrt{2}}\sqrt{\frac{16\int^{r_b}\Lambda(\tau)\tau\, d\tau + C}{r_b^4} - 1} \quad \ldots (15)$$

integration constant C is determined by requiring E_z to be continuous.

In the region before the beamload,

$$\lambda(\xi) = \Lambda(r_b) = 0 \qquad (0 \leq \xi \leq \xi_t)$$

Boundary condition: $\xi = 0 \Rightarrow r_b(\xi=0) = R_b \,, \quad r_b'(\xi=0) = 0$

$$\boxed{E_z(\xi) \approx -\frac{r_b}{2\sqrt{2}} \sqrt{\frac{R_b^4}{r_b^4} - 1}} \quad 0 \leq \xi \leq \xi_t \quad \dots (16)$$

The condition we seek is to have a current profile such that

$$E_z(r_b \leq r_t) = \frac{1}{2} r_b \left(\frac{dr_b}{d\xi}\right)_{r_b = r_t} \approx \text{constant} = -E_t$$

The continuity of electric at r_t requires that

$$\Rightarrow E_t = E_z(\xi_t)$$

$$\therefore E_z(\xi > \xi_t) = E_t = -\frac{r_t}{2\sqrt{2}} \sqrt{\frac{R_b^4}{r_t^4} - 1}$$

The shape of the bubble is described by parabola:

$$\frac{1}{2} r_b r_b' = E_t \quad \text{Boundary condition: } \xi = \xi_t \,, \; r_b = r_t$$

$$\Rightarrow r_b^2 = r_t^2 - 4E_t(\xi - \xi_t)$$

Once we have r_b, we can solve for λ directly from Eqn 13

$$\Rightarrow r_b \frac{d^2 r_b}{d\xi^2} + 2\left(\frac{dr_b}{d\xi}\right)^2 + 1 = \frac{4\lambda(\xi)}{r_b^2}$$

$$\lambda(\xi) = E_t^2 + \frac{r_b^2}{4} = \frac{R_b^4 + r_t^4}{8 r_t^2} - \sqrt{\frac{R_b^4 - r_t^4}{8 r_t^2}} \, (\xi - \xi_t) \quad \dots (17)$$

This expression can be written only in terms of Et instead of rt by solving r_t^2 in terms of Et

$$\lambda(\xi) = \sqrt{E_t^4 + \frac{R_b^4}{2^4}} - E_t(\xi - \xi_t) \quad \dots (18)$$

This is the equation for a trapezoid.

Maximum total charge

We can calculate the maximum total charge that corresponds to loading the fields to Et by assuming that the charge extends all the way tot the back of the bubble, where the sheath reaches the ξ axis.

use $$r_b^2 = r_t^2 - 4E_t(\xi - \xi_t)$$

$$r_b \to 0 \Rightarrow \xi - \xi_t = \Delta\xi_{tr} = \frac{r_t^2}{4E_t} \quad \dots (19)$$

Average charge per unit length for the trapezoidal bunch:

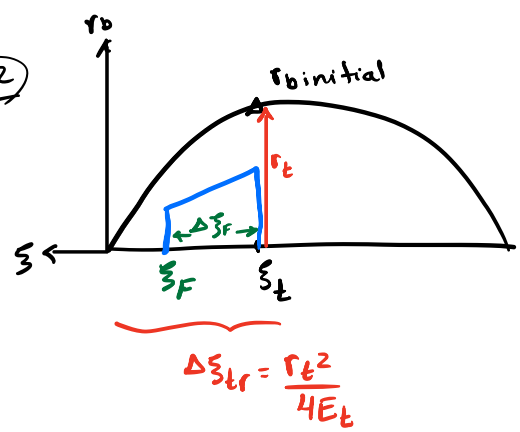
$$\langle\lambda\rangle_{\Delta tr} = \frac{\lambda(\xi_t + \Delta\xi_{tr}) + \lambda(\xi_t)}{2} = \frac{R_b^4}{8r_t^2} = \Lambda_0 \quad \dots (20)$$

$$Q_{tr} = 2\pi\Lambda_0\Delta\xi_{tr} = \boxed{\frac{\pi}{16}\frac{R_b^4}{E_t}} \quad \dots (21)$$

Beam loading efficiency of nonlinear beam loading

Let us start by assuming that beam loading terminated at some ξ_F, such that

$$\Delta\xi_F = \xi_F - \xi_t < \xi_{tr} \quad \dots (22)$$



For $\xi > \xi_F$, $\Lambda = 0$,

so the wakefield is described by the equation 22a with a new peak blowout radius $\tilde{R}_b$

$$E_z = -\frac{r_b}{2\sqrt{2}}\sqrt{\frac{\tilde{R}_b^4}{r_b^4} - 1}$$

By solving for this constant ($\tilde{R}_b$), we obtain the trajectory of the innermost electron behind the bunch, which allows us to determine the energy remaining in the wake. This constant can be found by using the continuity of the field at ξ_F:

$$
\begin{cases}
E_F = E_z(r_F) = \dfrac{-r_F}{2\sqrt{2}}\sqrt{\dfrac{\tilde{R}_b^4}{r_F^4} - 1} = E_t \\
r_F^2 = r_t^2 - 4E_t(\xi_F - \xi_t) \\
E_t = \dfrac{-r_t}{2\sqrt{2}}\sqrt{\dfrac{R_b^4}{r_t^4} - 1}
\end{cases}
$$

since E_z field is flattened starting at ξ_t

$$
\Rightarrow \boxed{\tilde{R}_b^4 = R_b^4\left(\frac{r_F^4}{R_b^4} + \frac{r_F^2}{r_t^2} - \frac{r_t^2 r_F^2}{R_b^4}\right)} \cdots (23)
$$

Energetics of the bubble in blowout regime

Consider eqn 13 in a region w/o beam:

$$
r_b r_b'' + 2r_b'^2 + 1 = 0
$$

Using the insight that $r_b'(\xi)\dfrac{dr_b'}{dr_b} = \cancel{r_b'}\dfrac{dr_b'}{d\xi}\cdot\dfrac{d\xi}{\cancel{dr_b}} = \dfrac{dr_b'}{d\xi} = r_b''$,

eqn 12 can be rearranged & integrated to give

$$
r_b^4(2r_b' + 1) = \text{Constant.}
$$

$\therefore$ we define a constant in the beam-free region given by

$$
I_0 = \frac{\pi}{16} r_b^4\left[1 + 2\left(\frac{dr_b}{d\xi}\right)^2\right] \cdots (24)
$$

Since we know at the location of max bubble radius, $r_b = R_b$ & $\dfrac{dr_b}{d\xi} = 0$, we can associate I_0 with a max bubble radius given by

$$
\Rightarrow \boxed{I_0 = \frac{\pi}{16} R_b^4} \cdots (25)
$$

Also note,

$$
\frac{dI_0}{d\xi} = \frac{\pi}{4} r_b^3 \frac{dr_b}{d\xi}\left[r_b\frac{d^2 r_b}{d\xi^2} + 2\left(\frac{dr_b}{d\xi}\right)^2 + 1\right]
$$

So when $\lambda \neq 0$, $$\frac{dI_0}{d\xi} = -\frac{\pi}{2} r_b^3 \frac{dr_b}{d\xi} \left[\frac{2\lambda(\xi)}{r_b^2} \right]$$

If we integrate this expression between ξ_0 & ξ, and using the Expression for $E_z = \frac{1}{2} r_b r_b'$, for a region of beam-loaded electrons, we get

$$I_0(\xi) - I_0(\xi_0) = -\underbrace{2\pi}_{=\int_0^{2\pi} d\phi} \int_{\xi_0}^{\xi} \int_0^{r_b} E_z(\xi') \rho_B(\xi', r') \, r' \, dr' \, d\xi'$$

The integral on the right hand side is essentially $q \int \vec{J}_B \cdot \vec{E} \, dV$, which is the power of the energy exchange between the field of the bubble & the electron bunch in the volume bounded by ξ_0 & ξ.

Recall that the relativistic energy conservation is given by

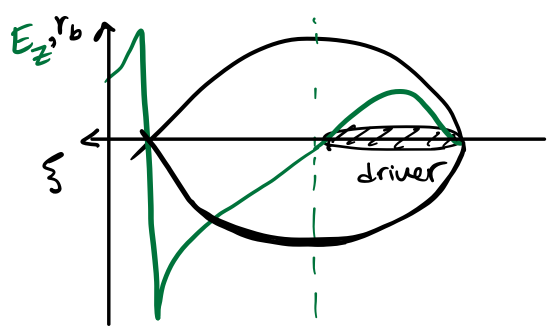
$\frac{d}{dt}(\gamma mc^2) = q(\vec{E} \cdot \vec{v})$ for a charge q

$dq = \rho \, dV \implies \int \vec{E} \cdot \vec{v} \, \rho \, dV$

$= \int \vec{E} \cdot \vec{J} \, dV$: rate of energy exchange for the bunch

Therefore, I_0 serves as a measure of energy density in the bubble.

As the drive bunch generates the wake, I_0 increases since in that region, $E_z > 0$. This means that the beam is pumping energy into the wake, resulting in correspondingly larger & Larger max



bubble radius, R_b. On the other hand, when the trailing beam is accelerated by the wake, $E<0$ so I_0 decreases. Therefore after the trailing beam, the trajectory of r_b would follow that corresponding to a bubble with $\tilde{R}_b < R_b$ since

$$I_0 = \frac{\pi}{16} R_b^4$$

Golovonov, 2021

$$\tilde{I}_0 = I_0 \text{ after the beam} < I_0$$

for the bubble before beamloading, I_0 represents the energy given to the wakefield by the driver per unit time. Behind the trailing bunch, I_0 drops to

$$I_0 = \pi/16 \, \tilde{R}_b^4,$$

which is the energy left in the wakefield. Therefore, the efficiency of beam loading, being the efficiency with which the trailing beam extracts energy is given by

$$\eta_b = 1 - \left(\frac{\tilde{R}_b}{R_b}\right)^4 \quad \ldots \quad (26)$$

This equation in principle holds for any bunch shape. For a flattened field, we can use equation 23 to find the efficiency of beam loading:

$$\eta_b = 1 - \left(\frac{r_F^4}{R_b^4} + \frac{r_F^2}{r_t^2} - \frac{r_t^2 r_F^2}{R_b^4}\right) \quad \ldots \quad (27)$$

Using the expression for Q_{tr} (Equation 21),

$$Q_{tr} = \frac{\pi R_b^4}{16 E_t} \quad \text{(maximum charge)}$$

and an equivalent expression for Q_F (bunch ends at ξ_F), we ge-

$$\boxed{\eta_b = \frac{Q_F}{Q_{tr}}} \quad \ldots \quad (28)$$

Note: as $\Delta\xi_F \to \Delta\xi_{tr}$

$$Q_F \to Q_{tr} \Rightarrow \eta_b \to 100\%$$

Simulation results

Here, we look at the simulation results using the parameters described in Tzoufras 2009:

$$n_b(r,\xi) = \frac{N_b}{(2\pi)^{3/2}\sigma_r^2\sigma_z} e^{-r^2/2\sigma_r^2} e^{-(\xi-\xi_c)^2/2\sigma_z^2}$$

$$\left.\begin{aligned} &\sigma_r = 0.5\ c/\omega_p, \quad N_b = 139\,(c/\omega_p)^3 \\ &\sigma_z = 1.414\ c/\omega_p, \quad \xi_c = 10\ c/\omega_p \end{aligned}\right\} \Rightarrow k_p R_b = 5$$

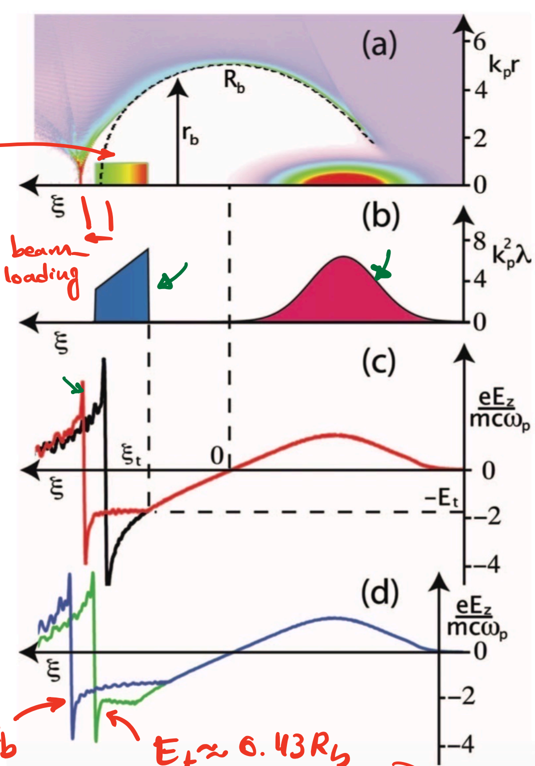
optimal trapezoid Loaded at $\xi_t = 18.2\ c/\omega_p$

observed from simulations $\longrightarrow$ $E_t = 0.35 R_b = 1.75\ c/\omega_p$

Calculated
$$\left\{\begin{aligned} &Q_{tr} \simeq 70/k_p^3 \quad \text{(loaded charge)} \\ &k_p^2\lambda = 7.0 - 1.75\times(\xi - 18.2) \quad \text{(optimal beam density)} \\ &\Delta\xi_{tr} = [\lambda(\xi_t) - E_t^2]/E_t = 2.2\ c/\omega_p \end{aligned}\right.$$

r_b without beam loading

elongation due to beam loading



$E_t \approx 0.25 R_b = 1.3\ c/\omega_p$

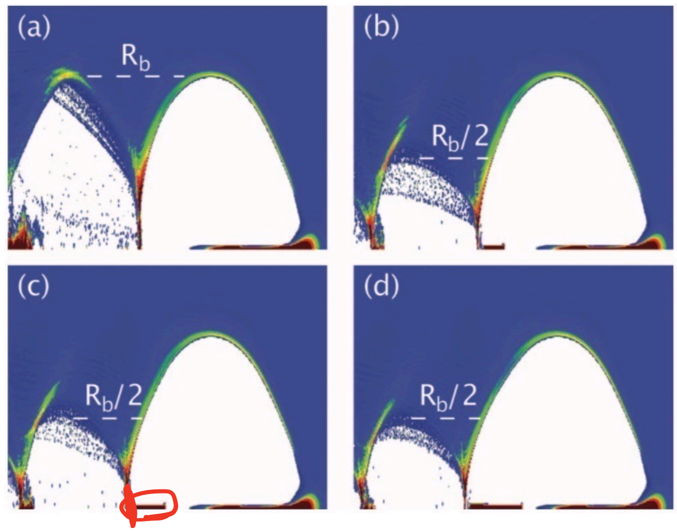
$E_t \approx 0.43 R_b = 2.2\ c/\omega_p$

For these latter cases the wakefield is not as flat. The reason for this is that kpRb is not large enough for Eq. 13 to be completely accurate. This illustrates the size of errors that may result if kpRb is not large enough. If the charge of the bunch is increased/ decreased slightly for blue/green cases, the wakes can then be made to be more flat. For very large blowout radii the differences between theory and simulation are negligible.

(a): no beam loading

(b): optimal trapezoidal bunch

(c)
(d) } trapezoidal bunch in different locations



The size of the blowout behind the beam loading is dropped to around 1/2. So the efficiency is

$$\eta_b = 1 - \left(\tfrac{1}{2}\right)^4 = 93.75\,\%$$

This is an extremely efficient process by accelerator physics standards!

Note that the similar Rb, and therefore efficiency for cases (b-d) confirms the theoretical prediction that, in contrast to linear regime, the efficiency is independent of the accelerating gradient.

<u>Dechirping</u>

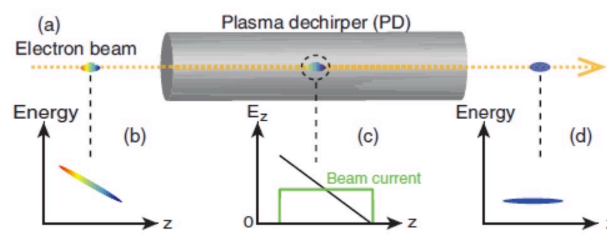
Over the last decade since Tzoufras proposed the ideal beam loading with a trapezoidal-shaped beam, the creation of such a beamload has remained an unsolved research problem. Another solution has emerged based on the fact that the beam loading of a flat-top or a Gaussian beam load results in a linear accelerating field. The electrons accelerated thus have a spatial “linear chirp” at the end of their acceleration length.

So rather than creating a situation for a flat electric field inside the bubble, several simulation and experimental groups have investigated the idea of dechirping, I.e. removing the linear chirp after the beam goes through its acceleration length. Below I will describe two such ideas commonly discussed in the community

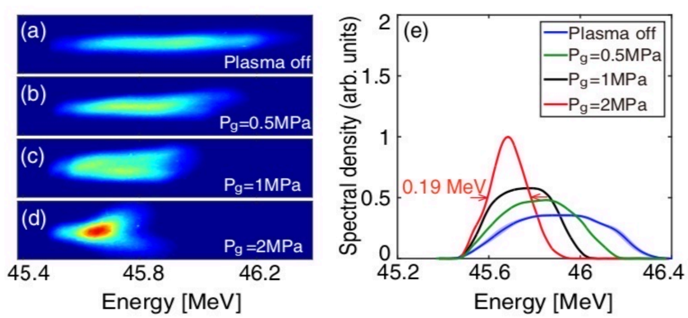
1. Sending the resulting beam through low density plasma (see e.g. Wu, et al., PRL, 204804, 2019)

The initially chirped beam is sent through a low density plasma and if the parameters of the plasma are chosen properly, the self fields generated in the plasma produce the opposite chirp resulting in a beam with almost no energy spread. Simulations suggest a 10x reduction in energy spread down to 0.1% is possible.

Schematic of the idea with the self fields in the plasma counteracting the initial energy spread.

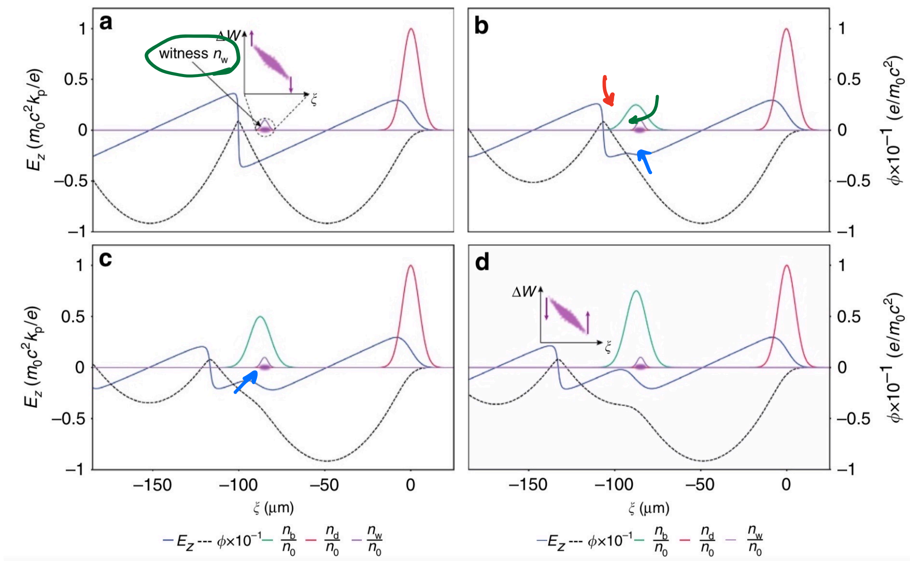


Experiments snd stimulants show that the energy spread can be significantly reduced.



2. Proper overloading of the wake by an escort beam (see e.g. Manahan, et al. Nature Communications 8, 15705, 2017)

The idea in this paper is to use a Gaussian and accelerate it until it has reached the desire energy. At that point, a second bunch can be injected to co-propagate with the initial beam, with its position and charge set such that C<0 in the position of the original beam. The opposite slope that is generated then will remove the energy spread from the initial beam.



The figure above shows how the slope of the accelerating field changes for different charges in the escort beam (all beams have Gaussian profiles). The idea is to transition from a case where the wakefield is 'under loaded' to a case where it is 'overloaded', I.e. where the slope of the field flips

Particle injection and trapping

One of the biggest challenges in the field is how to put particles inside the high-gradient accelerating field.

We start from the constant of motion obtained from the co-moving coordinate in regular SI (unnormalized) units:

$$\gamma mc^2 - P_z c + q(\phi - A_z c) = \text{constant}$$

This results was obtained with the co-moving coordinate defined as

$$\xi = ct - z$$

To investigate the dynamics of particles in a wakefield that is traveling at the phase velocity v_ϕ , it is more useful to define

$$\xi = v_\phi t - z \quad \dots \; 1$$

This is particularly useful in the case of a laser driven wakefield, since the phase velocity of wake is usually relatively small (typically, $\gamma_\phi \sim 10$)

Since the co-moving coordinate is keeping up with the wake, the quasi-static approximation is applied once again, and we obtain a constant of motion similar to the previous case, except that $c \rightarrow v_\phi$:

$$\gamma mc^2 - P_z v_\phi + q(\phi - A_z v_\phi) = \text{Constant} \quad \dots \; 2$$

This is the Hamiltonian for the system. This can be obtained by the canonical transformation of standard E&M Hamiltonian using a generating function

$$F_2 = P_0 x - v_p \int (P_z - eA_z)\, dt$$

for an e^- & in normalized units, Eqn 2 becomes

$$\gamma - P_z v_\phi - \psi = \text{const} = h_0 \quad \dots \; 3$$

initial Hamiltonian of the particle

We can use this relationship to study the possible trajectories in the phase space for the wake (Esirkepov, PRL 2008)

Since $\gamma^2 = 1 + P_\perp^2 + P_z^2$,

$$h_0 = \sqrt{1 + P_z^2 + P_\perp^2} - \psi(\xi) - v_\phi P_z$$

Each unique (and not crossing!) particle trajectory is defined in phase space by a particular h_0.

Rearranging for P_z,

$$P_z = \gamma_\phi^2 (h_0 + \psi(\xi)) \left\{ 1 \pm V_\phi \left[1 - \frac{1 + P_\perp^2}{\gamma_\phi^2 (h_0 + \psi)^2} \right]^{1/2} \right\} \quad \ldots (7)$$

Physical Solutions for P_z exist when

$$1 \geq \frac{1 + P_\perp^2}{\gamma_\phi^2 (h_0 + \psi)^2} \Rightarrow (h_0 + \psi)^2 \geq \frac{1}{\gamma_\phi^2} + \frac{P_\perp^2}{\gamma_\phi^2} \quad \ldots (8)$$

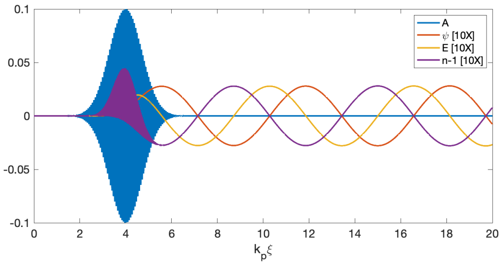
The +/- components join each other where

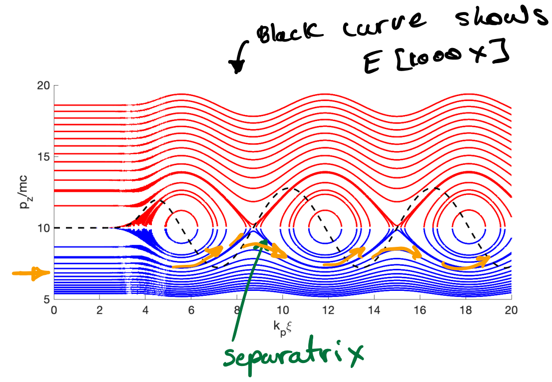
's' for separatrix

$$P_{z,s} = V_\phi \gamma_\phi (1 + P_\perp^2)^{1/2} \quad \ldots (9)$$

Let's take a look at the phase space ($P_z(\xi)$) for particles for a case where $\gamma_\phi = 10$ (see the Appendix for details regarding derivation of ψ & n_e)

e.g. for $\gamma_\phi \sim 10$, $a_0 \sim 0.1$,

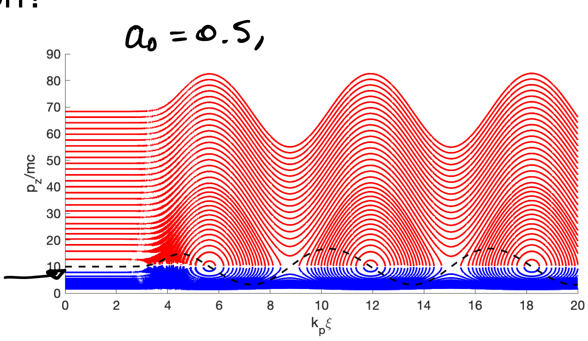




The red curve represents the positive solutions and the blue curves represent the negative solutions for Eqn 7. Note the following features:

- Trajectories that include $P_{z,s} = \gamma_\phi \sim 10$, will consist of closed orbits in phase space. Note that particles with these hamiltonians do not "travel" in the wake. Mathematically, this is because the expression under the square root in equation 7 is negative for a range of ψ, and so there is no physical solution for particles in that range with a particular hamiltonian. The physical interpretation is that these particles are trapped in the wake, and they move forward along the red curve and go backwards along the blue curve in the co-moving coordinate.
- Other trajectories that not include $P_{z,s}$ simply move forward (the red curves) or backwards (blue curves). The physical interpretation for these particles is that they either have too high a momentum or too low a momentum and simply move along the wake (forwards or backwards)
- The curve separating the trapped trajectories and the traveling trajectories is called the separatrix, and represents the last "traveling particle" that isn't trapped. This particle continually loses and gains energy but is just below the threshold of trapping.
- Note that the energy gain and loss is directly related to the electric field (the black dashed line in phase space image). Therefore, injection of prior electrons and beam loading, which modifies the electric field will distort the orbit for the electrons that haven't been trapped yet.

What happens for a more nonlinear situation? Now with the increased value of ψ, higher and higher momentum values can be reached for the trapped electrons as the larger field creates a larger accelerating field.



Trapping Condition

From the phase space discussion, we know a particle is on a trapped orbit if P_z from Eqn 7 has a physical solution inside the wake for $V \sim V_\phi$.

$$8 \Rightarrow (h_0+\psi)^2 \geq \frac{1}{\gamma_\phi^2} + \frac{P_\perp^2}{\gamma_\phi^2}$$

$$\boxed{|h_0+\psi| \geq \frac{\sqrt{1+P_\perp^2}}{\gamma_\phi}} \quad \ldots (16)$$

This is the 3D trapping condition.

Note that there is a second condition. If h_0 is so large, so that the inequality in eqn 16 is satisfied regardless of ψ, then that describes a travelling particle. Here, we require h_0 to be on the order of ψ, so that for part of the cycle the inequality 16 is not satisfied. Only then does the Hamiltonian "h_0" describe that of a trapped particle.

<u>Note</u>: Trapping threshold can also be obtained by using the eqn. of constant of motion for Hamiltonian & letting $V_z \to V_\phi$

$$\gamma(1 - V_\phi V_z) = h_0 + \psi$$

in 1D: $$\gamma_\phi (1 - V_\phi^2) = h_0 + \psi$$

$$\Rightarrow (h_0+\psi) = \frac{1}{\gamma_\phi} \quad \ldots (17)$$

Eqn 17 can be derived from 16 by setting $P_\perp \to 0$

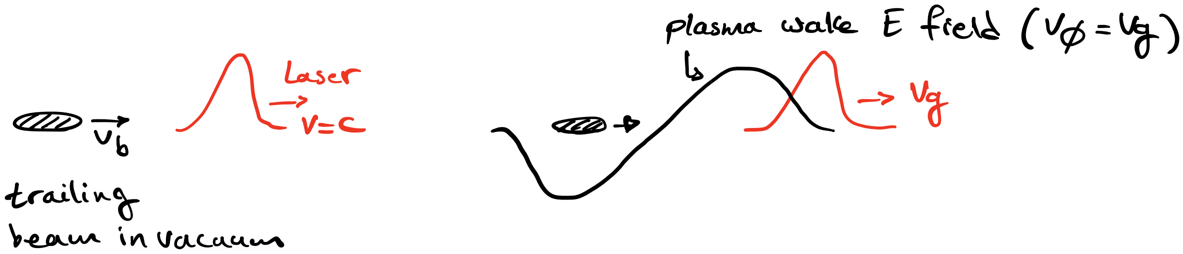
In 3D, need to consider transverse dynamics, i.e. e^- must not transversely escape the bubble before reaching β_ϕ.
Given the equation for the Hamiltonian, we can further simplify inequality 16:

$$\left.\begin{aligned} h_0 &= \gamma_i - P_{zi} - \Psi_i \\ 16 \Rightarrow |h_0 + \Psi| &\geq \frac{\sqrt{1+P_\perp^2}}{\gamma_\phi} \end{aligned}\right\} \Rightarrow |\Delta\Psi + (\gamma_i - P_{zi})| \geq \frac{\gamma_\perp}{\gamma_\phi} \quad \ldots (18)$$

Here, $\Delta\Psi = \Psi - \Psi_i$ represents the change from the particle's initial wake function. The various trapping mechanisms then can be described in terms of inequality 18.

Trapping Mechanisms

There are two general classes of solutions to the problem of injecting a trailing beam on a trapped orbit. The first class is called "external injection", where a trailing beam is prepared and sent together with the driver into the plasma. Once the plasma wake forms, the trailing beam finds itself on the trapped orbit and accelerates forward:



This is a difficult task because the small size of plasma wavelength means that the trailing beam has to have a short duration $\sigma_z \lesssim \frac{1}{2}\, c/\omega_p$, & be focused to an equally small size. Moreover, if the driver is Laser, the jitter (meaning shot to shot variation) of laser to e-beam has to be smaller than c/ω_p.

This is a tall order, even for the most advanced e-beam sources in the world, such as LCLS, which has a jitter ~ 20 fs. For an e-beam driver, this problem was circumvented by creating both driver & trailing from the same e-beam. This effectively elliminated jitter between drive & trailing beams & Lead to a successful external injection experiment. See Litos, Nature, 515, 92 (2014).

The trapping condition for this mechanism is stated as follows:

$$\gamma_i \ \& \ P_{zi} \gg 1 \quad \& \quad \gamma_\perp \ll \gamma, \gamma_\phi \quad \& \quad \psi_i = 0$$

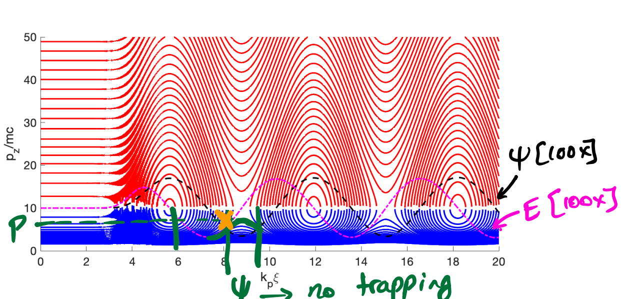
$$\boxed{\gamma^2 = 1 + P_z^2 + \cancel{P_\perp^2}}$$

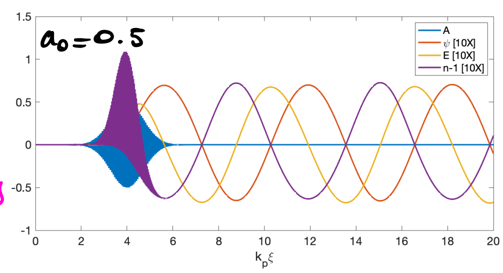
$$\gamma_i - P_{zi} = \gamma_i - \left(\gamma_i^2 - 1\right)^{1/2}$$

$$= \gamma_i - \gamma_i \left(1 - \frac{1}{\gamma_i^2}\right)^{1/2}$$

$$\gamma_i - P_{zi} = \frac{1}{2\gamma_i} \approx \frac{1}{2P_{zi}} \quad \ldots (19)$$

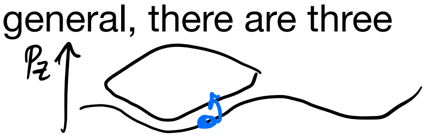
$$\Rightarrow 18 \Rightarrow \left|\psi + \frac{1}{2\gamma_i}\right| \geq 0 \quad \Rightarrow \quad \boxed{\psi \geq -\frac{1}{2\gamma_i}}$$





The physical interpretation is that for $\gamma_i < \gamma_\phi$, the electron may not be able to get trapped if placed at the wrong phase of the wake. In that case, it will be placed on a traveling "blue trajectory"

The second solution is to get background plasma electrons to transition from their regular passing orbits in phase space to the trapped orbits. By the way from the previous figure, you can see that if the plasma is warm enough, some electrons with $v \sim v_\phi$ will get trapped. In general, there are three strategies to facilitate this transition:

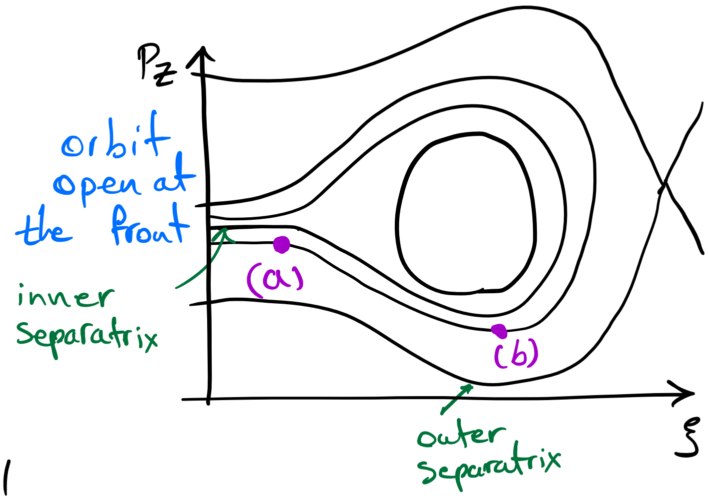


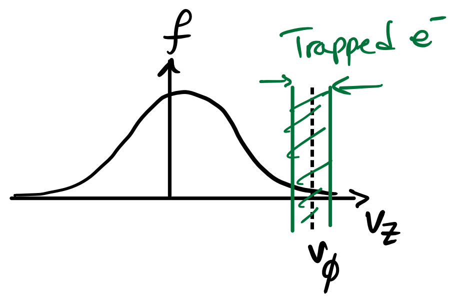
1. Initialize particles on trapped orbit

2. Sudden change in Hamiltonian
3. Drive wake to wave breaking or "self-injection" amplitude

① Initialize e^- on trapped orbit

Conceptually, this is the simplest strategy. The idea is to initialize e^- on an orbit that is already inside the seperatrix, i.e. point (a) or (b). Consider point (a). A particle in this place must have above average fluid momentum. In a thermal plasma at high temperature for example, Some e^- would have $v_z \approx v_\phi$. These e^- could get trapped & accelerated provided the satisfy inequality 18





$$\left|\Delta\Psi + (\gamma_i - P_{zi})\right| \geq \frac{\gamma_\perp}{\gamma_\phi}$$

for these e^-, $\Psi_i = 0$, use inequality 19 to simplify $\gamma_i - P_{zi}$

$$\gamma_i - P_{zi} = \frac{1}{2\gamma_i}$$

$$\Rightarrow \left|\Psi + \frac{1}{2\gamma_i}\right| \geq \frac{\gamma_\perp}{\gamma_\phi} \quad \ldots \quad (20)$$

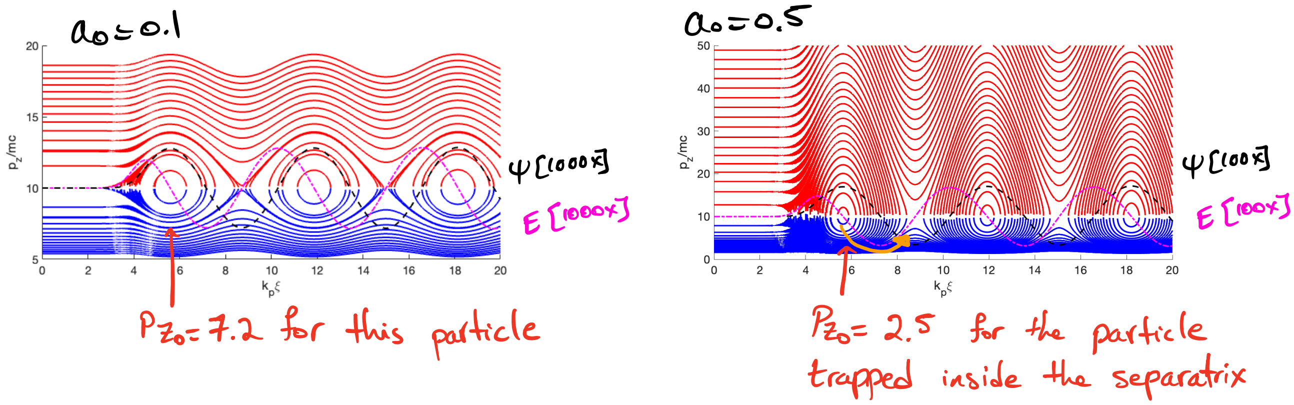
Since $\Psi < 0$ at the back of the back of the wake, the minimum γ_i that will result in trapping has to satisfy

$$\Psi_{min} + \frac{1}{2\gamma_i} \leq -\frac{\gamma_\perp}{\gamma_\phi}$$

$$\Rightarrow \frac{1}{2\gamma_i} \leq |\Psi_{min}| - \frac{\gamma_\perp}{\gamma_\phi}$$

$$\Rightarrow \boxed{\gamma_i \geq \frac{1}{2}\left[|\psi_{min}| - \frac{\gamma_\perp}{\gamma_\phi}\right]^{-1}} \quad \cdots (21)$$

At point (b), a particle is created within the wake (i.e. at $\psi_i \neq 0$). From the phase space plots, it can be observed that for such a particle to be trapped in the linear regime (low a_0), it still has to be generated w/ substantial forward momentum. In the nonlinear regime, e.g. $a_0 = 2$, this gap widens to an e^- with $P_{zi} = 0$ can even get trapped.



The physical interpretation is that as the electric field amplitude increases, an electron initiated inside the wake with lower and lower energy can gain enough energy from the wakefield to reach the phase velocity of the wake. From the trapping inequality,

$$|\Delta\psi + (\gamma_i - P_{zi})| \geq \frac{\gamma_\perp}{\gamma_\phi}$$

initiating an e^- w/ $P_{zi} \approx 0$, $\gamma_i \sim 1$ at ψ_i,

$$|\Delta\psi + 1| \geq \frac{\gamma_\perp}{\gamma_\phi}$$

Since $\psi_{min} < 0$, minimum ψ_i for trapping is reached for

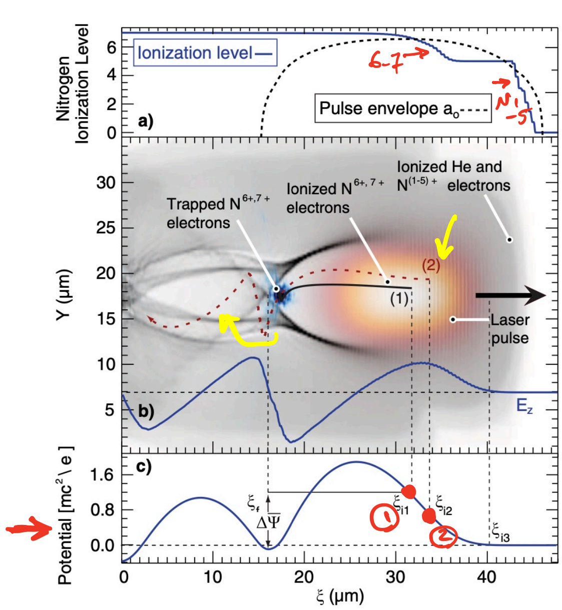
$$\Delta\psi + 1 \leq -\frac{\gamma_\perp}{\gamma_\phi}$$

$$\Delta\psi \leq -1 - \frac{\gamma_\perp}{\gamma_\phi} \quad \cdots (22)$$

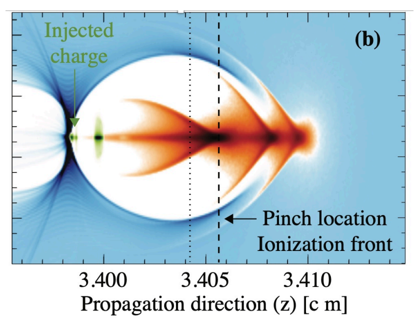
If the transverse momentum is small, $\gamma_\perp \ll \gamma_\phi$,

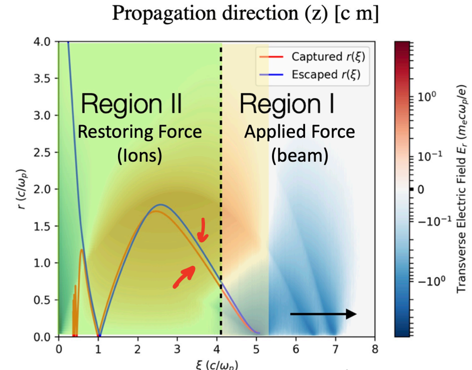
$$\boxed{\Delta\Psi \leq -1} \quad \cdots (23)$$

This is the condition for ionization injection, where an electron is ionized inside a wakefield at correct phase leading to its trapping. This phenomenon was first observed in electron beam experiments by Oz, et al. PRL, 2006 and by A. Pak and C. McGuffy, PRL 2010 (two back to back articles). In these experiments an inner electron shell is ionized either at the peak of the laser pulse, or at the focused point of an oscillating drive electron beam.



A. Pak, 2010





N. Vafaei-Najafabadi, 2019

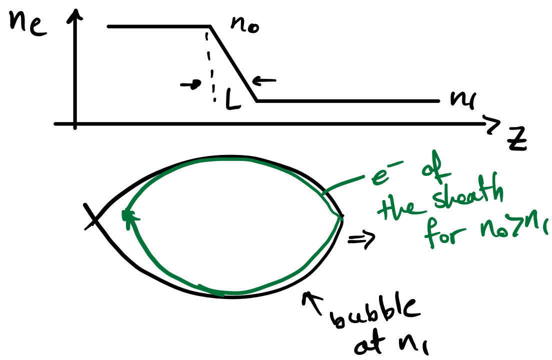
J. Yan, unpublished

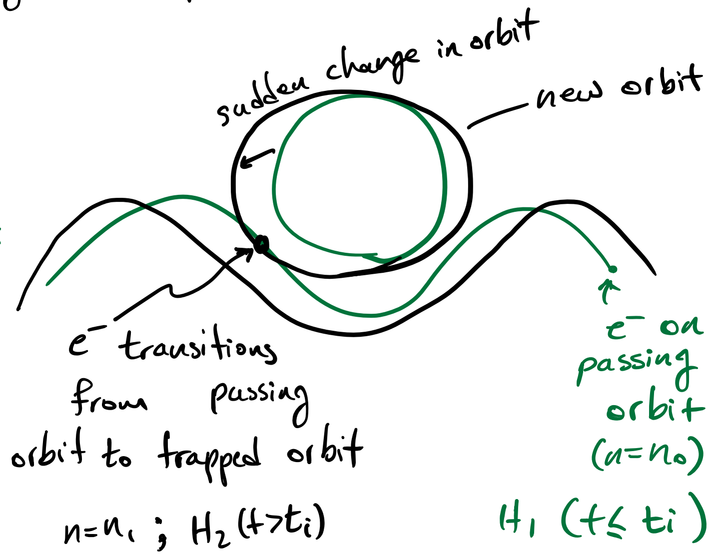
II Sudden Change in Hamiltonian

A sudden change in the Hamiltonian can shift the particles in phase space from a travelling orbit to a trapped orbit. These changes occur on the scale of the wake formation itself, which means that the quasi-static approximation is violated & the Hamiltonian is no longer a constant of motion.

Examples:

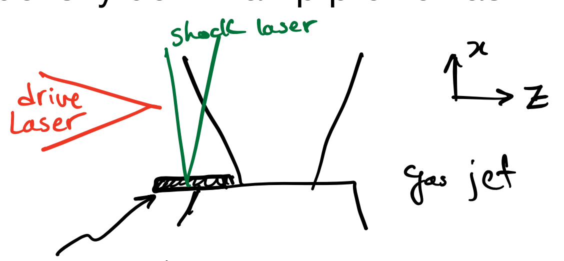
(a) sharp density downramp (also called shock injection): a change in density of plasma with a scalength comparable to c/ω_p would displace e^- from a travelling to trapped orbit:

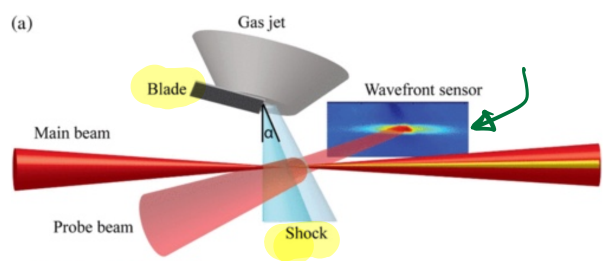




See e.g. Suk PRL 2001, Bulanov PRL 1998, Geddes PRL 2009 Scmidt PRSTAB 2012, Xu PRAB 2017

Experimentally, the most recent effort in laser wakefield to produce this density down ramp profile has involved creating a density shock



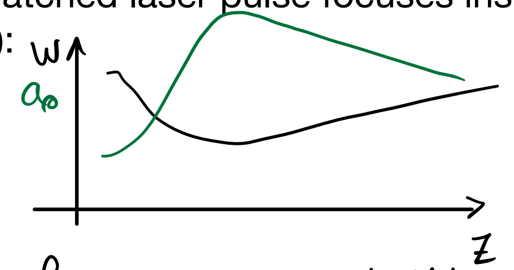


Laser-solid interaction creates a plasma plume moving in $+x$ direction. Adjusting the delay allows the driver laser to interact with shock+ gasjet (pioneered by NRL)

D. Kaganovich, J. Appl. Phys. 116, 013304 (2014)

Another method includes partial obstruction of the gasjet, which results in a shock density profile, see e.g. K.K. Swanson, PRAB 2017

Rapid elongation of the wakefield can also naturally occur in an LWFA as a mismatched laser pulse focuses inside the plasma (see e.g. Kostyukov PRL 2009):



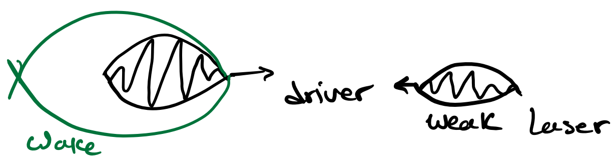
$W \downarrow$

$a_0 \uparrow$

$k_p R_b \uparrow$

Laser focusing causes bubble expansion → change in Hamiltonian.

(b) Colliding pulse injection: use a second laser to modify the wake structure at a time scale on the order of ω_p^{-1}



colliding pulses set up a standing wave & modify particle orbits. see e.g. Faure Nature 2006 & Malka, PoP 2009

In general, any phenomenon that interferes with the ordinary trajectory of electrons forming the wake can lead to injection. The most commonly observed injection method in experiments is still the natural evolution of the high-amplitude plasma wake which leads to injection.

III Driving wake to wavebreaking amplitude "self injection"

These e^- start at rest in front of the wake.

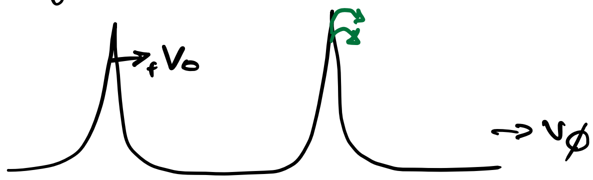
For these particles, $h_0 = \gamma - P_z - \psi = 1$ & trapping threshold is

$$\Rightarrow \boxed{\psi = \frac{1}{\gamma_\phi} - 1} \quad \dots (24)$$

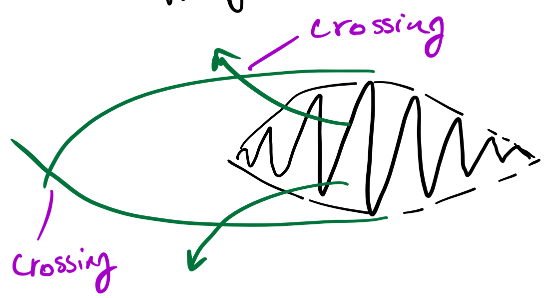
The catch is that e^- have to make it to a region where $\psi \approx -1$. This is an impossibly large value for the wake function even for highly nonlinear wakes at $a_0 \sim 4$. For these particles to get trapped then, the regular structure of the wake must "break", leading to injection of some of the background e^-

In 1D, we already saw "wave breaking" is the limit where fluid velocity v_0 approaches v_ϕ.

The wave breaks like an ocean wave & some particles roll over into the trapped orbits.



In 3D, we can have trajectory crossing w/o trapping:



But, much like the previous section, temporal variation in wake structure, can push the particles (particularly at the back of the wake structure) to become trapped.

Simulations have been used to get an empirical predictor for conditions to achieve trapping by Benedetti PoP 2013:

$$a_0(\gamma_\phi) \simeq 2.75 \left(1 + \left(\frac{\gamma_\phi}{22}\right)^2\right)^{1/2}$$

Appendix 1, a discussion on γ_ϕ

It is useful to divide this discussion to the case of a beam driver and laser driver. This is because beam drivers are highly relativistic:

$$E = \gamma m c^2$$

for e^-, $mc^2 = 0.511$ MeV

$\Rightarrow$

E	γ
1 GeV	~ 2000
10 GeV	$\sim 20{,}000$

In contrast, a laser driver is modestly relativistic. We can get an estimate for the group velocity of a laser using our earlier study of linear plasma waves:

$$\beta_g^2 = \left(\frac{v_g}{c}\right)^2 = 1 - \frac{\omega_p^2}{\omega_0^2}$$

$$\gamma_g = \left(\frac{1}{1-\beta_g^2}\right)^{1/2} = \frac{\omega_0}{\omega_p}$$

e.g. for a $1\mu m$ laser ($\omega_0 = 1.88\times10^{15}$ Hz) & $n_0 = 1\times10^{18}\ cm^{-3}$ ($\omega_p = 5.63\times10^{13}$), $\boxed{\gamma_g \sim 33}$. The nonlinear effects of high laser amplitude tend to increase γ_g, while 3D effects reduce it. For a more thorough discussion, see Esarey, Review of modern physics, 81, 1229 (2009), but the upshot is that the plasma wakefield having an equal phase velocity to the group velocity of the laser, will have a $\gamma \sim O(10)$, which means that it can be outrun by e^- of modest energy $\sim O(100$ MeV$)$ $\gamma \sim 200$.

What about β? $\beta^2 = 1 - \frac{1}{\gamma^2} \Rightarrow \beta =$

	$\gamma = 10$	$\gamma = 30$
β	0.995	0.9994

$\beta \approx 1$ is satisfied to a very good approximation.

In what follows we will look at motion of particles in wake assuming that wake now moves w/ $\gamma_\phi = \gamma_g$

Appendix 2: Differential eqn connecting ψ to A in 1D:

$$\frac{\partial^2\psi}{\partial\xi^2} + \frac{1}{2}\left[1-\frac{1+A^2}{(1+\psi)^2}\right]=0 \quad \cdots (10)$$

note: $P_\perp = A$ → canonical momentum

The assumption used to derive this equation was $v_\phi \approx c$, $\gamma_\phi \gg 1$. One can derive a more exact expression for a wake with any γ_ϕ, but even for the case of $\gamma_\phi \sim 10$, Eqn 10 is accurate to $O(\gamma^{-2}) \sim 1\%$. Using a "wide" laser pulse with a Gaussian profile allows us to solve eqn 10 in 1D, i.e. $\frac{\partial^2\psi}{\partial\xi^2} \rightarrow \frac{d^2\psi}{d\xi^2}$ & $\nabla_\perp^2 \rightarrow 0$. For the laser,

$$A = a_0 \cos(k_0\xi)\, e^{-(\xi-\xi_0)^2/\tau^2} \quad \cdots (11)$$

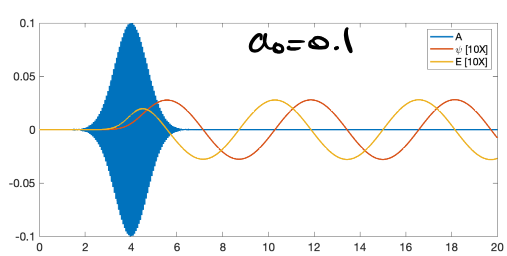
Figures below show the wake function ψ & density perturbation $(n-1)$ generated by this laser. "n-1" is calculated using the expressions (see "Laser & beam coupling to plasma" notes):

$$\frac{n}{\gamma} = \frac{1+\cancel{\nabla_\perp^2\psi}}{1+\psi} \quad \cdots (12)$$

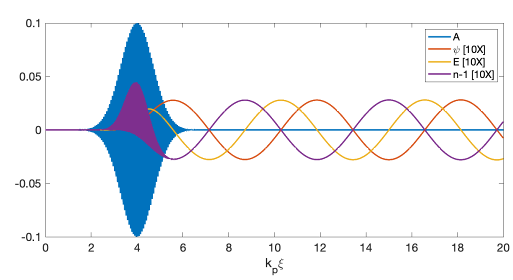
$\nabla_\perp^2\psi = 0$, 1D analysis

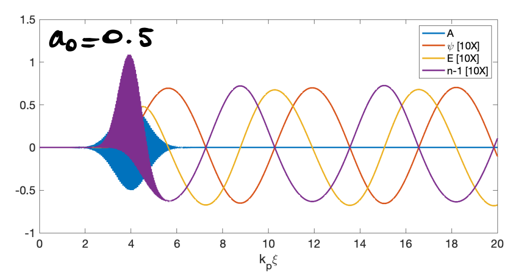
$P_\perp^2 = A^2$

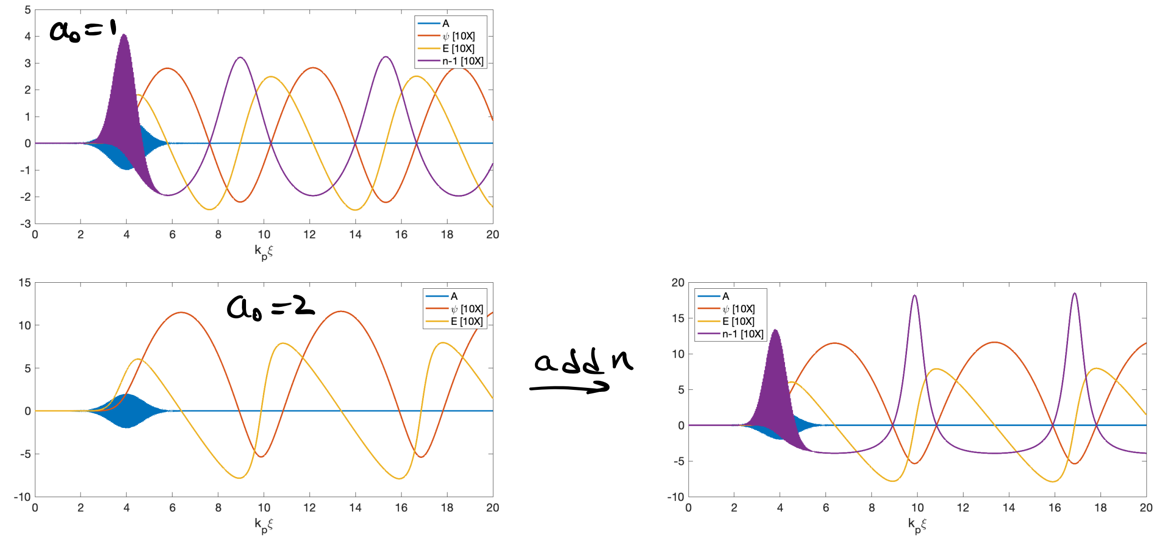
$$\gamma = \left(\frac{1+P_\perp^2+(1+\psi)^2}{2(1+\psi)}\right) \quad \cdots (13)$$



add n →







Note several features of these results

1. The nonlinear plasma wavelength increases with laser strength
2. In the region of the laser, density oscillates (zooming in, you can see that the density oscillations are at the second harmonic).
3. For the wake function and the electric field in the region of the laser, the oscillations are only a small perturbation on these functions.

Having obtained the wake functions, we can initialize a number of electrons with different Hamiltonians (h_o), and using Eqn 7, look at their behavior in the phase space, P_z vs. ξ.

To make the physical meaning of eqn. 8 clear, we choose a set of Hamiltonian values that result in $P_{zs} = \beta_\phi \gamma_\phi$ in the region w/o laser, i.e. $A=0$. Consider the case of

's' for separatrix

$\gamma_\phi \sim 10 \Rightarrow \beta_\phi = 0.995$, $\quad P_{zs} \sim \gamma_\phi \sim 10$

$$⑥ \xRightarrow{A=0,\psi=0} \quad h_o = \sqrt{1+P_z^2} - v_\phi P_z$$

$v_\phi = \left(1 - \frac{1}{\gamma_\phi^2}\right)^{1/2}$

$$h_o = P_z \left(1 + \frac{1}{P_z^2}\right)^{1/2} - v_\phi P_z$$

$P_z >> 1$, $\gamma_\phi^2 >> 1$

$$\approx P_z \left(1 + \frac{1}{2P_z^2}\right) - P_z \left(1 - \frac{1}{2\gamma_\phi^2}\right)$$

$$h_o \approx \frac{P_z}{2}\left(\frac{1}{P_z^2} + \frac{1}{\gamma_\phi^2}\right) \quad \ldots (14)$$

$P_{z,s} = \gamma_\phi$

$$\therefore \boxed{h_{os} \sim 1/P_{zs}} \quad \ldots (15)$$

Some References (this is not an exhaustive list of work in this area, but is only meant to serve as a starting point for literature research)

- E. Adli, et al,, “Design of a TeV Beam Driven Plasma-wakefield Linear Collider”, Proc. IPAC 2013, TUPME020
- A. A. Golovonov, et, al., “excitation of strongly nonlinear plasma wakefield by electron bunches”, Plasma Phys. Contr. Fusion, 63, 085004 (2021)
- T. Katsouleas, et al., “beam loading in plasma accelerators”, Part. Accel., 22, 81 (1987]
- W. Lu, et al., “a nonlinear theory for multi-dimensional relativistic plasma wave wakefields”, Phys. Plasmas, 13, 056709 (2006)
- G. G. Manahan, et al. “Single-stage plasma-based correlated energy spread compensation for ultra high 6D brightness electron beams” Nat. Comm., 8, 15705, 2017
- M. Tzoufras, et al., “Beam loading by electrons in nonlinear plasma wakes”, Phys. Plasmas, 16, 056705 (2009)
- Y. Wu, et al., “Phase space dynamics of a plasma wakefield dechirped for energy spread reduction”, Phys. Rev. Lett., 204804, 2019

Particle Trapping and injection

- C. Benedetti, et al., Numerical investigation of electron self-injection in the nonlinear bubble regime, Phys. Plasmas 20, 103108 (2013);
- T. Esirkepov, et al, “Electron, Positron, and Photon Wakefield Acceleration: Trapping, Wake Overtaking, and Ponderomotive Acceleration”, Phys. Rev. Lett., 96, 014803 (2006)
- J. Faure, et al., Controlled injection and acceleration of electrons in plasma wakefields by colliding laser pulses, Nature 444, 737–739 (2006)
- D. Kaganovich, et al., Shaping gas jet plasma density profile by laser generated shock waves, J. Appl. Phys. 116, 013304 (2014)
- V. Malka et al., Laser-driven accelerators by colliding pulses injection: A review of simulation and experimental results, Phys. Plasmas 16, 056703 (2009)
- C. Mcguffey, et al., “Ionization Induced Trapping in a Laser Wakefield Accelerator”. Phys. Rev. Lett., 104:025004, 2010.
- E. Oz, et al. “Ionization-Induced Electron Trapping in Ultrarelativistic Plasma Wakes”. Phys. Rev. Lett., 98, 084801, 2007.
- A. Pak, et al. “Injection and Trapping of Tunnel-Ionized Electrons into Laser-Produced Wakes”. Phys. Rev. Lett., 104:025003, 2010.
- K. K. Swanson, et al., Control of tunable, monoenergetic laser-plasma-accelerated electron beams using a shock-induced density downramp injector, Phys. Rev. Accel. Beams 20, 051301 (2017).