A Toy Model of Numerical Cherenkov Radiation

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AAC 2022

Introduction

- Numerical Cherenkov radiation can occur when the numerical phase velocity is reduced below c, allowing for resonance with the particle velocity
- In macro-particle models the resonant response of numerical Cherenkov radiation is associated with an instability
- In a system equivalent to using canonical coordinates, we wish to confirm that even when numerical Cherenkov radiation leads to exact resonance, there is no resulting instability
  - By analyzing a sufficiently simple toy model which captures the behavior of numerical Cherenkov radiation, we can produce analytically the same solution as the computer would solve
  - So long as no exponential growth finds its way into the solution, we can be confident our model doesn't suffer from a numerical Cherenkov instability

The Toy Model

We consider a two-dimensional system with a single macro-particle having *fixed* velocity $\mathbf{u} = u\hat{\mathbf{z}}$ and an arbitrary shape function S.

$$\rho = q\, S(\mathbf{r} - \mathbf{u}\, t)$$

The Lagrangian is then

$$\mathcal{L} = \frac{-mc^2}{\gamma} + \frac{1}{8\pi}\int_{\Omega} \mathrm{d}V \left\{ 8\pi q\, S(\mathbf{r} - \mathbf{u}\, t)\left(\frac{\mathbf{u}}{c}\cdot\mathbf{A} - \varphi\right) + \frac{1}{c^2}\dot{\mathbf{A}}^2 + \frac{2}{c}\dot{\mathbf{A}}\cdot\nabla\varphi + (\nabla\varphi)^2 - (\nabla\times\mathbf{A})^2 \right\}$$

and only has $\mathbf{A}$ and φ as variable fields.

The Spacial Grid

The domain is limited to an L_x by L_z rectangle with N_x by N_z grid points

- Scalar fields restricted to the grid are represented as matrices
- The integral is replaced by a tensor M representing the quadrature rule
- The vector calculus operations are replaced with finite difference tensors

The discrete Lagrangian is then

$$\mathcal{L} = \frac{-mc^2}{\gamma} + \frac{1}{8\pi}\left\{ \frac{1}{c^2}\dot{\mathbf{A}}^{\mathsf{T}} M \dot{\mathbf{A}} + \frac{2}{c}\dot{\mathbf{A}}^{\mathsf{T}} M\, \mathbf{grad}(\varphi) + \mathbf{grad}(\varphi)^{\mathsf{T}} M\, \mathbf{grad}(\varphi) \right.$$

$$\left. - \mathbf{curl}(\mathbf{A})^{\mathsf{T}} M\, \mathbf{curl}(\mathbf{A}) + 8\pi \frac{\mathbf{u}}{c}^{\mathsf{T}} q\, S\, M\, \mathbf{A} - 8\pi q\, S\, M\, \varphi \right\}$$

Discrete Equations of Motion

Varying the action yields the equations of motion

$$\frac{1}{c^2}\ddot{\mathbf{A}} + \frac{1}{c}\,\mathbf{grad}(\dot{\varphi}) + \mathbf{grad}(\mathrm{div}(\mathbf{A})) - \mathrm{div}(\mathbf{grad}(\mathbf{A})) - 4\pi\,\frac{\mathbf{u}}{c}\,q\,S = 0$$

$$\frac{1}{c}\,\mathrm{div}(\dot{\mathbf{A}}) + \mathrm{div}(\mathbf{grad}(\varphi)) - 4\pi q\,S = 0$$

and the light cone gauge $A_{\mathbf{n}_z} = \varphi_{\mathbf{n}}$ can be used to limit the three equations of motion to two

$$\frac{1}{c^2}\ddot{A}_x + \frac{1}{c}\,D_x\dot{A}_z + D_x D_z A_z - D_z^2 A_x = 0$$

$$\frac{1}{c^2}\ddot{A}_z + \frac{2}{c}\,D_z\dot{A}_z + \frac{1}{c}\,D_x\dot{A}_x + D_x D_z A_x + D_z^2 A_z = 4\pi\left(1 + \frac{u}{c}\right)q\,S$$

Simplify Equations of Motion with Discrete Fourier Analysis

The discrete Fourier decomposition

$$f_{\mathbf{n}} = \frac{1}{N_x N_z} \sum_{\ell_x=0}^{N_x-1} \sum_{\ell_z=0}^{N_z-1} e^{i(n_x \Delta x\, k_x + n_z \Delta z\, k_z)} f_{\mathbf{k}} \quad ; \qquad k_x = \frac{2\pi\ell_x}{L_x} \quad , \quad k_z = \frac{2\pi\ell_z}{L_z}$$

can be used to replace central finite difference operations as

$$\mathcal{F}[\{\nabla f_{\mathbf{n}}\}]_{\mathbf{k}} = i\,\boldsymbol{\theta}_{\mathbf{k}} f_{\mathbf{k}} \qquad \text{for some } \textit{real } \boldsymbol{\theta}_{\mathbf{k}}$$

- For example, the second-order central difference

$$D_x f_{\mathbf{n}} = \frac{f_{n_x+1,n_z} - f_{n_x-1,n_z}}{2\Delta x} \quad \longrightarrow \quad \theta_{\mathbf{k}_x} = \frac{\sin(\Delta x\, k_x)}{\Delta x}$$

- Note that in the limit $\Delta x \to 0$, we must have that $\theta_{\mathbf{k}_x} \to k_x$

Discretized Equations of Motion in New Basis

The two matrix equations

$$\frac{1}{c^2}\ddot{A}_x + \frac{1}{c}D_x\dot{A}_z + D_xD_zA_z - D_z^2A_x = 0$$

$$\frac{1}{c^2}\ddot{A}_z + \frac{2}{c}D_z\dot{A}_z + \frac{1}{c}D_x\dot{A}_x + D_xD_zA_x + D_z^2A_z = 4\pi\left(1 + \frac{u}{c}\right)q\,S$$

can then be expressed mode-by-mode as

$$\ddot{A}_{\mathbf{k}_x} + ic\theta_{\mathbf{k}_x}\dot{A}_{\mathbf{k}_z} - c^2\theta_{\mathbf{k}_x}\theta_{\mathbf{k}_z}A_{\mathbf{k}_z} + c^2\theta_{\mathbf{k}_z}^2A_{\mathbf{k}_x} = 0$$

$$\ddot{A}_{\mathbf{k}_z} + 2ic\theta_{\mathbf{k}_z}\dot{A}_{\mathbf{k}_z} + ic\theta_{\mathbf{k}_x}\dot{A}_{\mathbf{k}_x} - c^2\theta_{\mathbf{k}_x}\theta_{\mathbf{k}_z}A_{\mathbf{k}_x} - c^2\theta_{\mathbf{k}_z}^2A_{\mathbf{k}_z} = Q_{\mathbf{k}}e^{-ik_zut}$$

where $Q_{\mathbf{k}}$ simply collects constants and is dependent on the macro-particle's shape function.

The Linear Algebra Problem

By introducing frequencies as $\omega_{\mathbf{k}_x} = c\theta_{\mathbf{k}_x}$ and $\omega_{\mathbf{k}_z} = c\theta_{\mathbf{k}_z}$, the previous equations can be written as the first-order system

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{bmatrix}\omega_{\mathbf{k}_x}A_{\mathbf{k}_x}\\ \dot{A}_{\mathbf{k}_x}\\ \omega_{\mathbf{k}_z}A_{\mathbf{k}_z}\\ \dot{A}_{\mathbf{k}_z}\end{bmatrix} = \begin{bmatrix}0 & \omega_{\mathbf{k}_x} & 0 & 0\\ \frac{-\omega_{\mathbf{k}_z}^2}{\omega_{\mathbf{k}_x}} & 0 & \omega_{\mathbf{k}_x} & -i\omega_{\mathbf{k}_x}\\ 0 & 0 & 0 & \omega_{\mathbf{k}_z}\\ \omega_{\mathbf{k}_z} & -i\omega_{\mathbf{k}_x} & \omega_{\mathbf{k}_z} & -2i\omega_{\mathbf{k}_z}\end{bmatrix}\begin{bmatrix}\omega_{\mathbf{k}_x}A_{\mathbf{k}_x}\\ \dot{A}_{\mathbf{k}_x}\\ \omega_{\mathbf{k}_z}A_{\mathbf{k}_z}\\ \dot{A}_{\mathbf{k}_z}\end{bmatrix} + \begin{bmatrix}0\\ 0\\ 0\\ Q_{\mathbf{k}}\end{bmatrix} e^{-ik_z ut}$$

which can be simplified to

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{y}_{\mathbf{k}} = M_{\mathbf{k}}\mathbf{y}_{\mathbf{k}} + \mathbf{V}_{\mathbf{k}} e^{-ik_z ut}$$

Discretized Time - Difference Equation

After diagonalizing we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{y}'_{\mathbf{k}} = D_{\mathbf{k}}\mathbf{y}'_{\mathbf{k}} + \mathbf{V}'_{\mathbf{k}}e^{-ik_z ut}$$

Solving this numerically leads to the corresponding difference equation

$$\mathbf{y}'^{(n+1)}_{\mathbf{k}} = \mathcal{D}_{\mathbf{k}}\mathbf{y}'^{(n)}_{\mathbf{k}} + \mathcal{V}_{\mathbf{k}}\mathbf{V}'_{\mathbf{k}}e^{-i\Lambda_{\mathbf{k}}n}$$

where $\Lambda_{\mathbf{k}} = k_z u\Delta t$ and $\mathcal{D}_{\mathbf{k}}$ and $\mathcal{V}_{\mathbf{k}}$ are diagonal matrices whose constructions depend on the time-stepping method employed. This can be solved exactly.

- Note that $\Lambda_{\mathbf{k}}$ is real-valued for all $\mathbf{k}$

The Eigenvalue Problem

Since the eigenvectors, $\mathbf{e}_i$, of $\mathcal{D}_\mathbf{k}$ span the space, we can define the eigenvalue problem as

$$\mathcal{D}_\mathbf{k}\mathbf{e}_i = e^{-i\Omega_\mathbf{k}^{(i)}}\mathbf{e}_i$$

and expand our vectors in terms of the eigenvectors

$$\mathbf{y}_\mathbf{k}^{\prime(n)} = a_\mathbf{k}^n\mathbf{e}_1 + b_\mathbf{k}^n\mathbf{e}_2 + c_\mathbf{k}^n\mathbf{e}_3 + d_\mathbf{k}^n\mathbf{e}_4$$

$$\mathcal{V}_\mathbf{k}\mathbf{V}_\mathbf{k}^\prime = \alpha_\mathbf{k}\mathbf{e}_1 + \beta_\mathbf{k}\mathbf{e}_2 + \gamma_\mathbf{k}\mathbf{e}_3 + \delta_\mathbf{k}\mathbf{e}_4$$

to solve for each vector component of the vector difference equation independently as

$$a_\mathbf{k}^{(n+1)} = a_\mathbf{k}^n e^{-i\Omega_\mathbf{k}^{(1)}} + \alpha_\mathbf{k} e^{-i\Lambda_\mathbf{k} n}$$

The Solution

The solution to the driven difference equation has two possible forms of solution depending on if $\Lambda_{\mathbf{k}} = \Omega_{\mathbf{k}}^{(1)}$.

$$a_{\mathbf{k}}^{n} = \begin{cases} A_{\mathbf{k}} e^{-i\Omega_{\mathbf{k}}^{(1)} n} + \alpha_{\mathbf{k}} \dfrac{e^{-i\Lambda_{\mathbf{k}} n}}{e^{-i\Lambda_{\mathbf{k}}} - e^{-i\Omega_{\mathbf{k}}^{(1)}}} & ; \quad \Omega_{\mathbf{k}}^{(1)} \neq \Lambda_{\mathbf{k}} \\ a_{\mathbf{k}}^{n} = \left(A_{\mathbf{k}} + \alpha_{\mathbf{k}} e^{i\Lambda_{\mathbf{k}} \Delta t} n\right) e^{-i\Lambda_{\mathbf{k}} n \Delta t} & ; \quad \Omega_{\mathbf{k}}^{(1)} = \Lambda_{\mathbf{k}} \end{cases}$$

The constant $A_{\mathbf{k}}$ is set by the initial conditions. The other three components are of identical form.

- This is qualitatively *identical* to the continuous solution to a driven oscillator

Accessibility of Solutions

Which of the two forms of the solution are obtainable is dependent on the numerical frequencies produced by the chosen time-stepping method.

- For example, with the implicit midpoint rule we have

$$\mathcal{D}_{\mathbf{k}} = \left(I - \frac{1}{2}\Delta t D_{\mathbf{k}}\right)^{-1}\left(I + \frac{1}{2}\Delta t D_{\mathbf{k}}\right)$$

  which results in a system with purely real values for $\Omega_{\mathbf{k}}^{(i)}$
    - Thus the resonant case of the result is achievable with this time-stepping method
- If we were to instead use the explicit RK-4 method

$$\mathcal{D}_{\mathbf{k}} = I + \Delta t D_{\mathbf{k}} + \frac{\Delta t^2}{2} D_{\mathbf{k}}^2 + \frac{\Delta t^3}{6} D_{\mathbf{k}}^3 + \frac{\Delta t^4}{24} D_{\mathbf{k}}^4$$

  we produce complex values for $\Omega_{\mathbf{k}}^{(i)}$, which can never meet the resonance condition.
    - If $\Omega_{\mathbf{k}}^{(i)} = \Omega_R + i\gamma$ then $\Omega_{\mathbf{k}}^{(1)} \neq \Lambda_{\mathbf{k}}$ must always be true

Conclusions and Future Work

- The resonant case of the solution corresponds to the potentials' numerical phase velocity in vacuum being reduced to the particle velocity
  - We note that any time-stepping method that introduces complex frequencies *cannot* produce this resonance as $\Lambda_{\mathbf{k}}$ must be real
  - The numerical phase velocity depends on the choice of finite difference; preliminary results show a clever choice of $\boldsymbol{\theta}_{\mathbf{k}}$ can lessen the effect of numerical Cherenkov radiation over certain ranges of $\mathbf{k}$
- The resonant case exhibits growth that is only *linear* in time and picks up a small phase shift proportional to the size of the time step
- If the toy model were improved to be fully self-consistent, energy considerations make it clear that the system could only drop out of resonance as the particle loses energy
  - This feature is to be shown when the toy model is expanded to self-consistent plasmas

Acknowledgements

I would like to thank my advisor Brad Shadwick and conference roommate Roland Hesse for discussions on this work and help in getting this presentation prepared.

This work supported by NSF grant numbers PHY-1535678 and PHY-2108788.