Thermal Modeling and Benchmarking of Crystalline Laser Amplifiers

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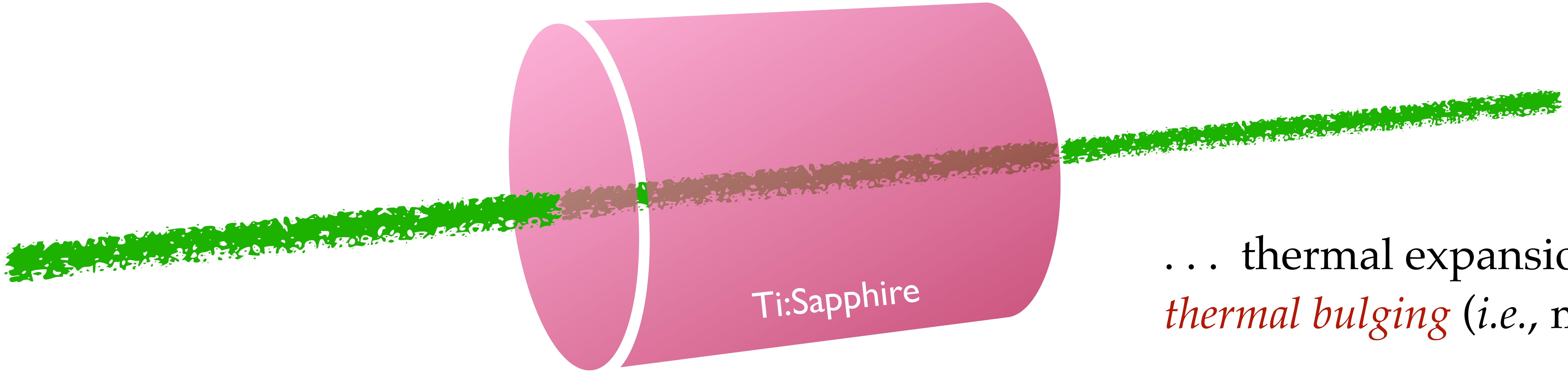
8 November 2022

AAC’22 Advanced Accelerator Concepts Workshop

This work is supported by the US Department of Energy, Office of Science, Office of High Energy Physics
under Award Numbers DE-SC0020931 and DE-AC02-05CH11231.

Motivation: Why Model Thermal Effects in Crystalline Laser Amplifiers?

At high power and high intensity, thermal gradients induce ...



. . . positional variation of the refractive index, hence *thermal lensing*

. . ., with direction-dependent thermal conductivity, hence *astigmatism*

. . . thermal expansion, hence *thermal bulging* (*i.e.*, more lensing)

. . . mechanical stresses, hence *modified birefringence*

Conservation of Energy and Fourier Heat Law yield (Nonlinear) Heat Equation

conservation of energy: $\rho \dot{u} = -\nabla \cdot \vec{q} + \dot{\varepsilon}(\vec{r}, t)$

mass density

internal energy per unit mass

heat flux

source term

Fourier law: $\vec{q} = -\kappa \, \nabla \theta$

thermal conductivity

temperature

specific heat capacity: $c_p = \dfrac{\partial u}{\partial \theta}$

relates gradients of energy and temperature: $\nabla u = \dfrac{\partial u}{\partial \theta} \nabla \theta = c_p(\theta) \, \nabla \theta \quad \Longrightarrow \quad \vec{q} = -\dfrac{\kappa(\theta)}{c_p(\theta)} \nabla u$

(nonlinear) heat equation:

$$\dot{u} = -\frac{1}{\rho} \nabla \cdot \vec{q} + \frac{1}{\rho} \dot{\varepsilon}(\vec{r}, t) = \nabla \cdot \Bigg(\underbrace{\frac{\kappa(\theta)}{\rho(\theta) c_p(\theta)}}_{\alpha(\theta) \to \alpha(u)} \nabla u \Bigg) + \frac{\kappa}{(\rho c_p)^2} \frac{\partial \rho}{\partial \theta} |\nabla u|^2 + \frac{1}{\rho} \dot{\varepsilon}(\vec{r}, t)$$

thermal diffusivity

neglect this term if density has a very weak dependence on temperature

linear heat equation:

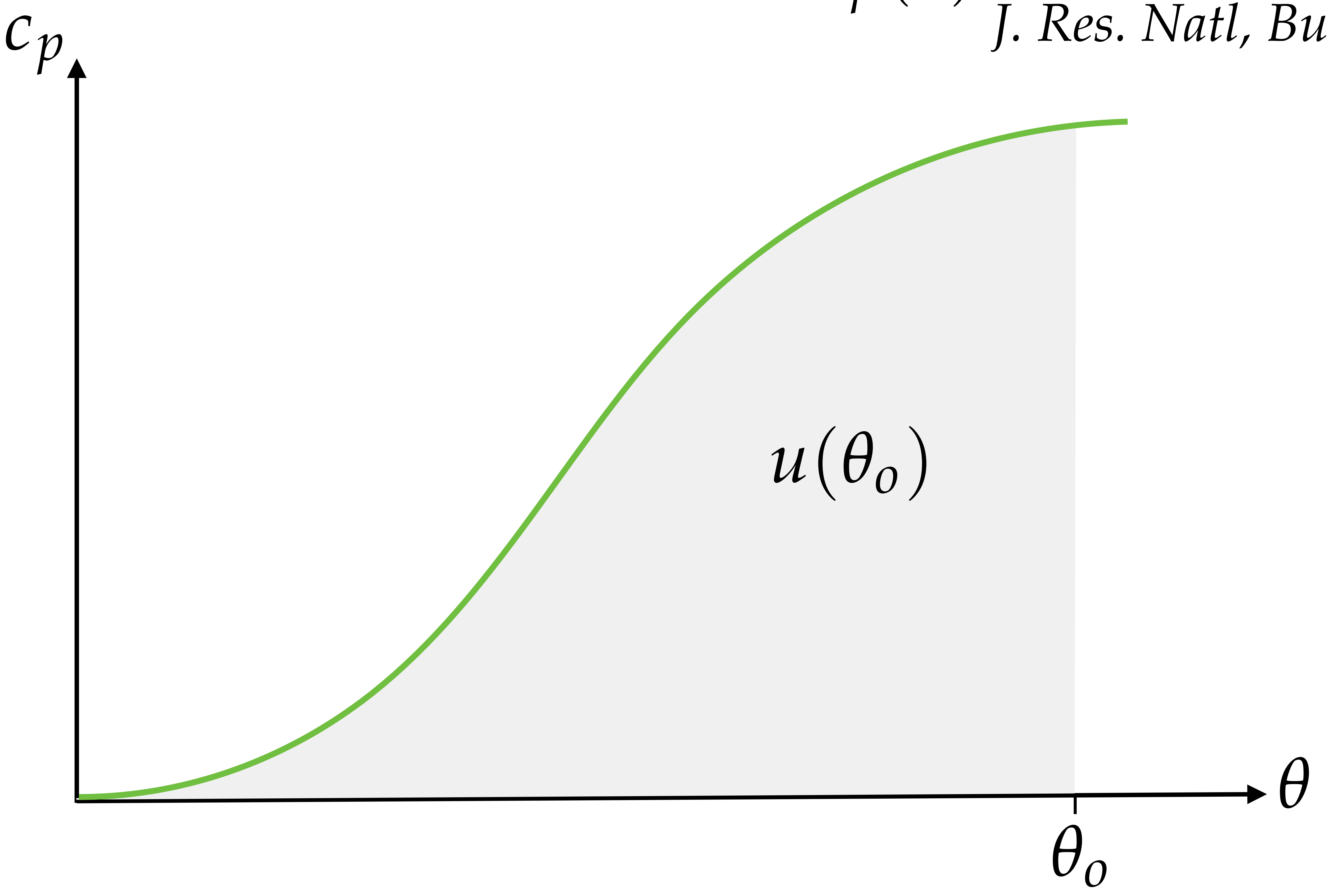
$$\boxed{\dot{u} = \alpha \, \nabla^2 u + \frac{1}{\rho} \dot{\varepsilon}(\vec{r}, t)}$$

Specific Heat Capacity Connects Internal Energy and Temperature

specific heat capacity: $c_p = \dfrac{\partial u}{\partial \theta}$

$\kappa(\theta)$ ER Dobrovinskaya, LA Lytvynov, and V Pishchik, "Properties of Sapphire", in *Sapphire: Material, Manufacturing, Applications*, (Springer, Boston, MA) 55–176, January 2009.
DOI: 10.1007/978-0-387-85695-7_2

$c_p(\theta)$ DA Ditmars, S Ishihara, SS Chang, G Bernstein, and ED West, *J. Res. Natl, Bur. Std.* 87(2):159–163, March-April 1982.



$$\Longrightarrow \quad \begin{aligned} \kappa(\theta) &\to \kappa(u) \\ c_p(\theta) &\to c_p(u) \\ \frac{\kappa(\theta)}{\rho(\theta)c_p(\theta)} &\to \alpha(u) \end{aligned}$$

NB: Sapphire $\neq$ Ti:Sapphire

Our Present Focus: Thermal Equilibrium with a Gaussian Pump Source

Time Scales for Thermal Relaxation (*linear* heat equation)

Fourier-Bessel series expansion

$$\Theta(r,z;t) = \sum_{n=1}^{\infty}\sum_{m=0}^{\infty} \Theta_{nm} J_0\Big(\frac{\nu_{0n} r}{a}\Big)\cos\Big(\frac{m\pi z}{l}\Big)\mathrm{e}^{-t/\tau_{nm}}$$

$$\tau_{nm}^{-1} = \alpha\Big[\Big(\frac{\nu_{0n}}{a}\Big)^2 + \Big(\frac{m\pi}{l}\Big)^2\Big]$$

dominant time-scale: $\tau_{10} = \dfrac{a^2}{\nu_{01}^2\alpha}$

Power Deposition and Equilibrium Temperature (*linear* heat equation)

transverse gaussian
× longitudinal decay

$$g(r,z) = g_0\, g_\perp(r)\, g_\parallel(z) = g_0\, \mathrm{e}^{-(r/\sigma)^2}\mathrm{e}^{-z/\zeta} \longleftarrow \dot{\varepsilon}(\vec{r},t)$$

$$\Theta(r,z) = \Theta_0 \sum_{n=1}^{\infty}\sum_{m=0}^{\infty} (X_{nm} R_n Z_m) J_0\Big(\frac{\nu_{0n} r}{a}\Big)\cos\Big(\frac{m\pi z}{l}\Big)$$

$$\Theta_0 = \frac{P_{\mathrm{abs}}}{4\pi K_c}\,\frac{V_{\mathrm{cyl}}}{V_{\mathrm{eff}}}$$

$$V_{\mathrm{eff}} = \int \mathrm{e}^{-(r/\sigma)^2}\mathrm{e}^{-z/\zeta}\,\mathrm{d}^3\vec{r} = \pi\sigma^2\zeta\big(1 - \mathrm{e}^{-\ell/\zeta}\big)$$

$$R_n = \frac{2}{[\nu_{0n} J_1(\nu_{0n})]^2}\int_0^{\nu_{0n}} x \exp\Big[-\Big(\frac{ax}{\nu_{0n}\sigma}\Big)^2\Big] J_0(x)\,\mathrm{d}x \qquad Z_m = \frac{2}{1+\delta_{m0}}\,\frac{l/\zeta}{(l/\zeta)^2 + (m\pi)^2}\big[1 - (-1)^m \mathrm{e}^{-l/\zeta}\big]$$

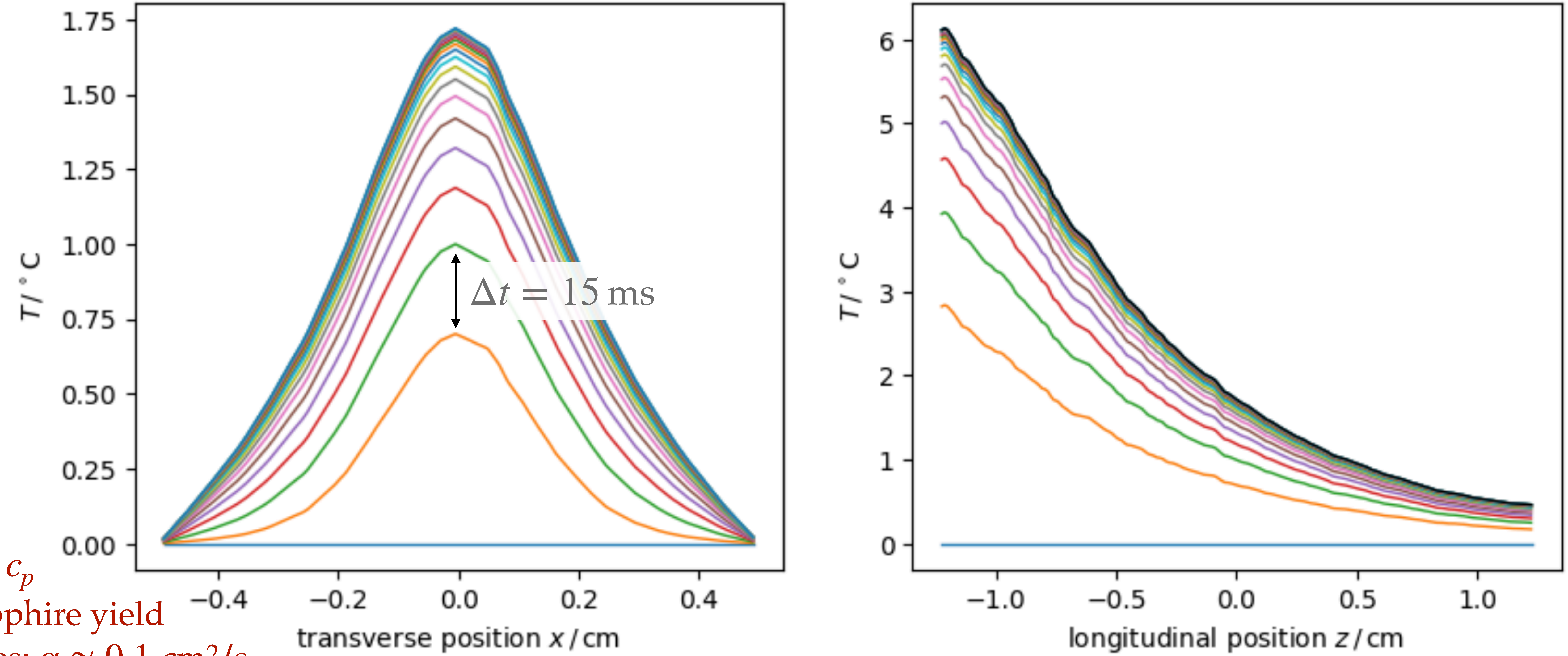
"cross"-term (from solving heat equation with vanishing time dependence) → $X_{nm} = \dfrac{l/a}{(\nu_{0n} l/a)^2 + (m\pi)^2}$

Initial Test with Linear Heat Equation: Quantitative Agreement and a Caveat

Cylindrical crystal:
radius $a = 1$ cm
length $\ell = 2.5$ cm
$\kappa = 0.31$ W/cm·K
$\rho = 3.98$ gm/cm^3
$c_p = 0.267$ J/gm·K
$\alpha = 0.292$ cm^2/s

$$\tau_{10} = \frac{a^2}{\nu_{01}^2 \alpha}$$

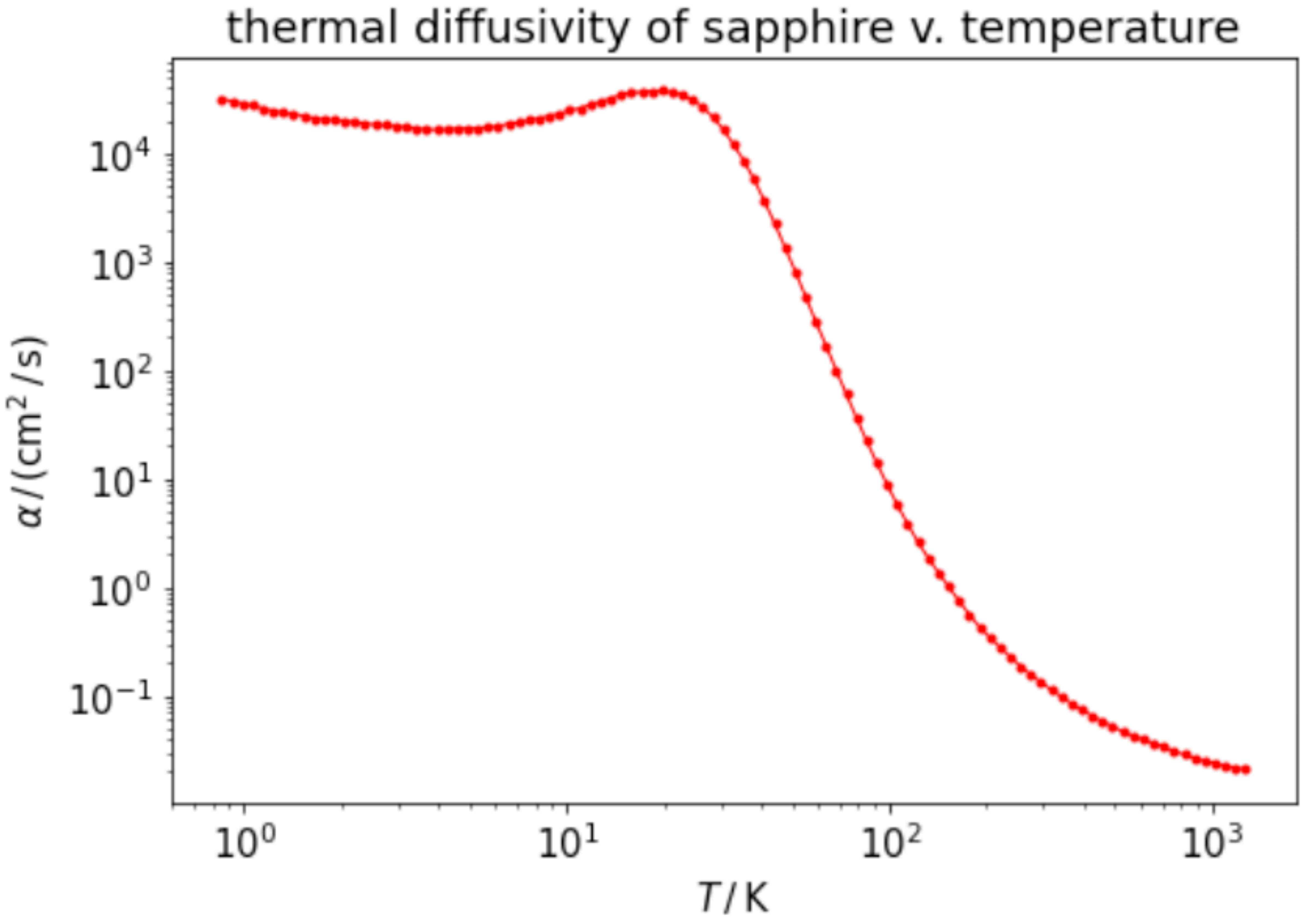
$\tau_{10} \approx 150$ ms $\Rightarrow$
$\alpha \approx 0.288$ cm^2/s
but values of κ and c_p tabulated for Ti:sapphire yield much smaller values: $\alpha \approx 0.1$ cm^2/s.

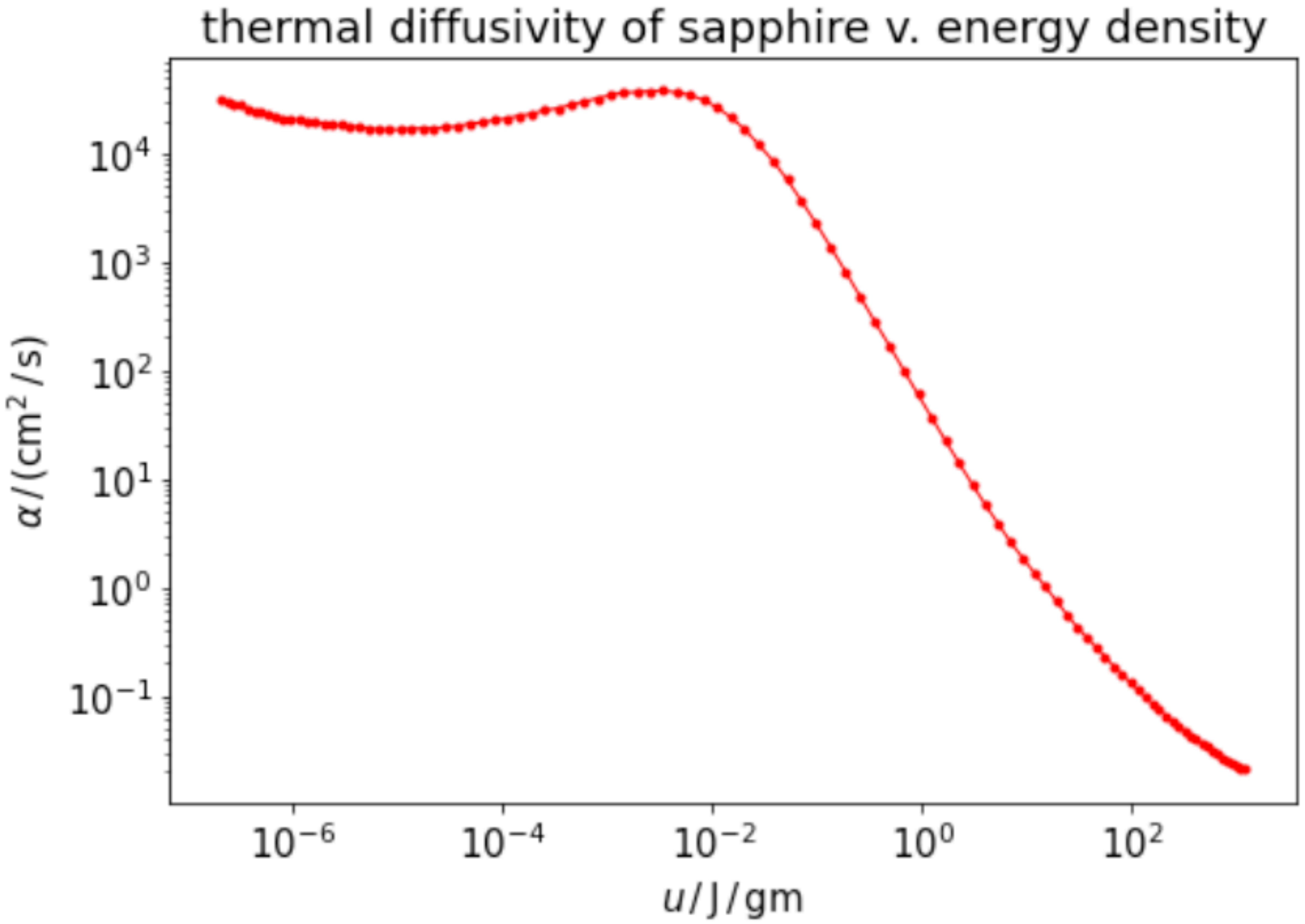


For numerical simulations of general PDEs, we use the open-source *FEniCS Computing Platform* (https://fenicsproject.org).
It uses UFL, the Unified Form Language (DOI: 10.48550/arXiv.1211.4047), to represent PDEs in variational form.
For details of simulation (and theory), see D.T. Abell, *et al.*, in *Proc. IPAC'22*, DOI: 10.18429/JACoW-IPAC2022-THPOTK062.

Thermal Diffusivity of Sapphire

On the basis of tabulated and graphical thermodynamic data for sapphire, we constructed the following plots for thermal diffusivity as a function of temperature (left) or internal energy density (right).





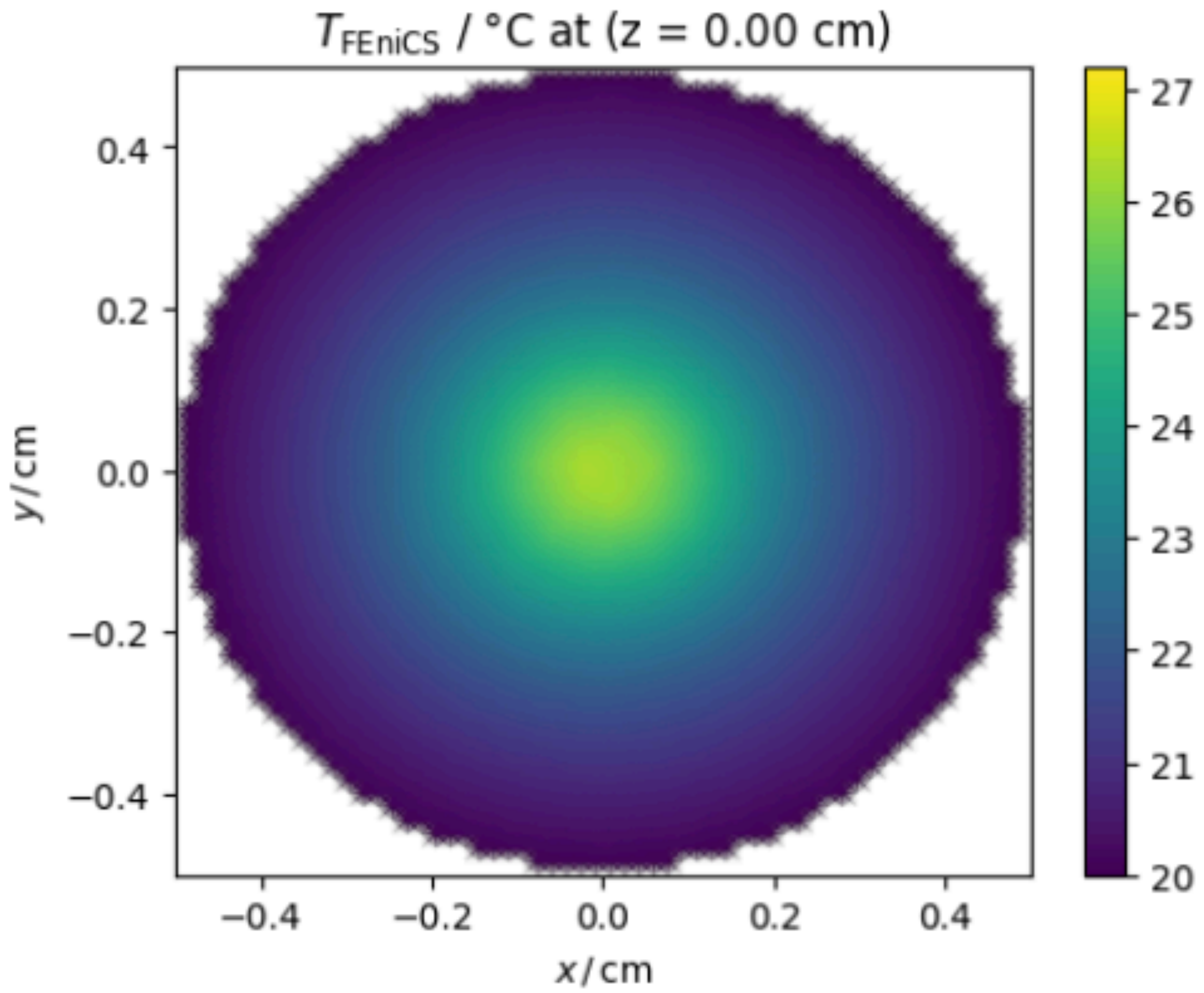
again … NB: Sapphire $\neq$ Ti:Sapphire

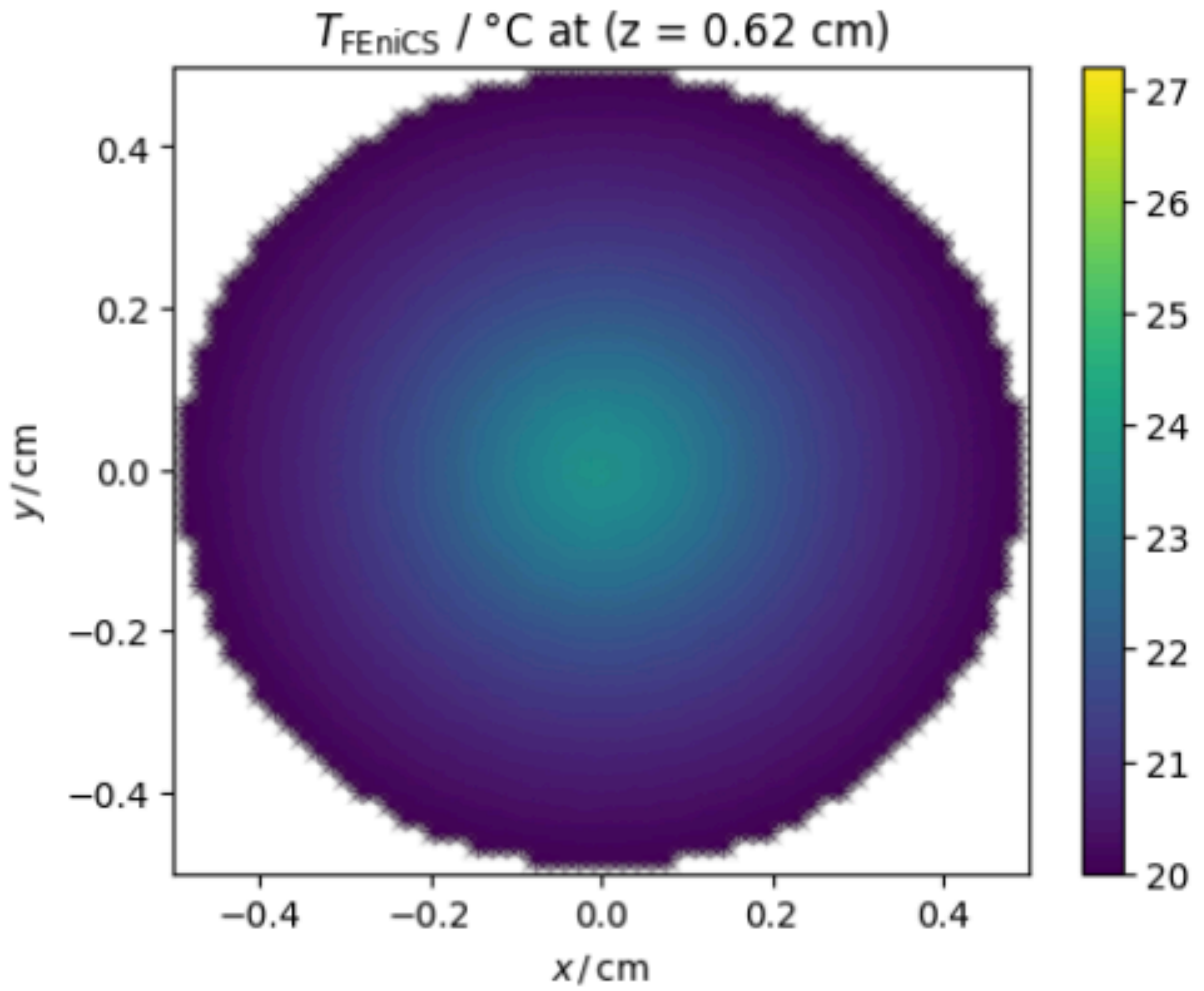
Nonlinear Poisson Equation in Three Dimensions

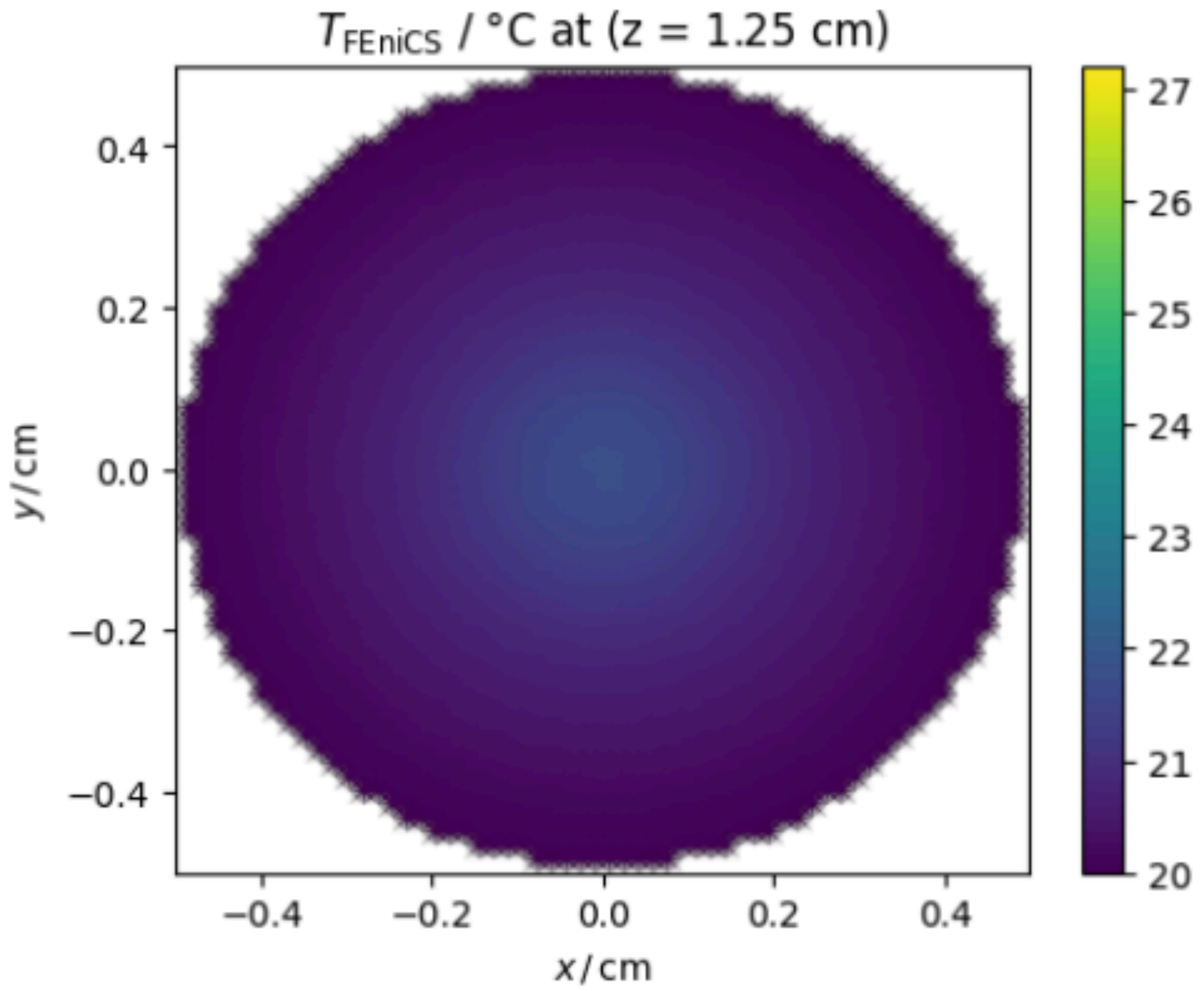
- Near room temperature, these results appear nearly identical to those of the linear case. (Temperature lineouts appear graphically identical.)

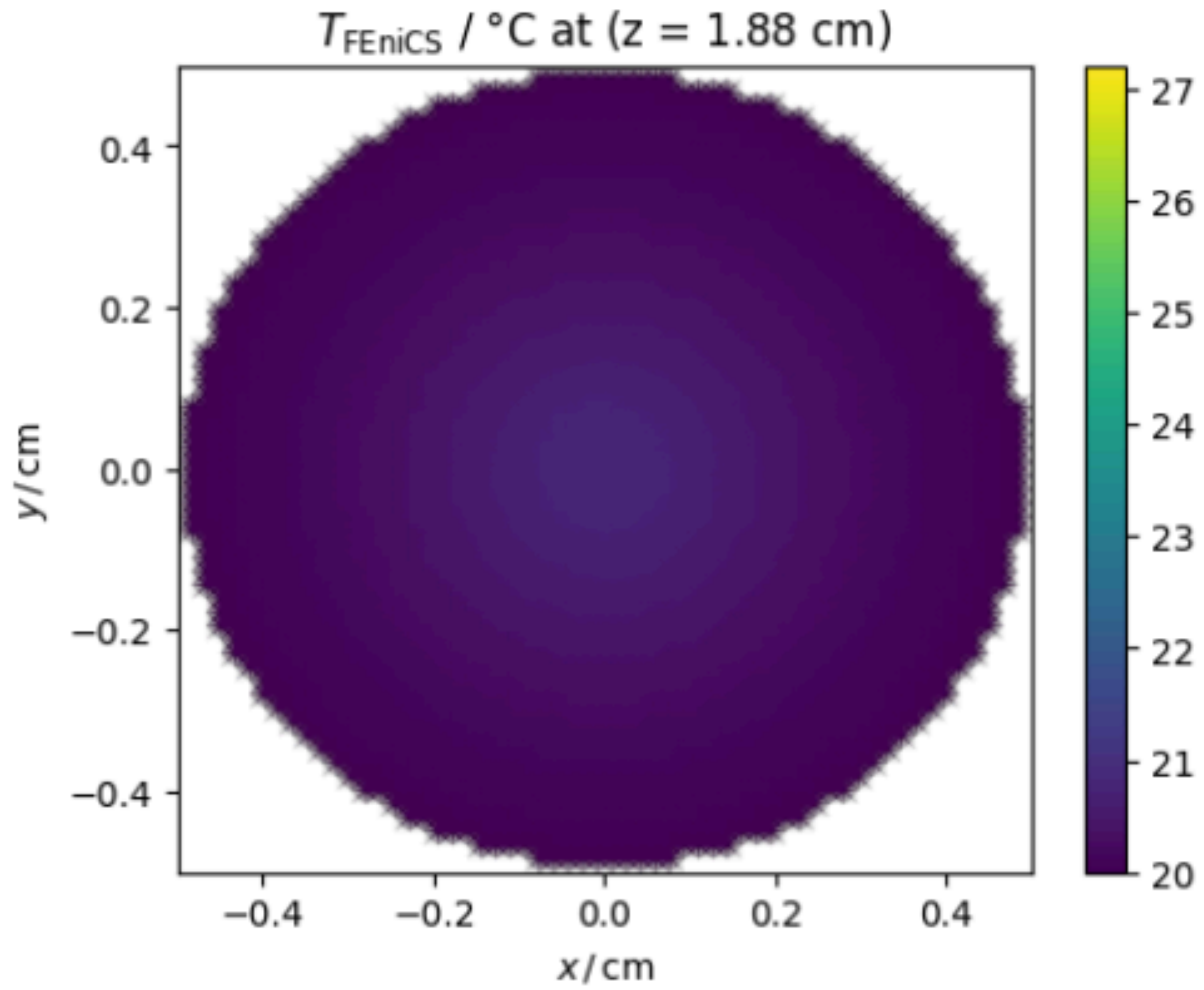
- At an × 5 power level—but still room temperature—we see only mild difference between the linear and nonlinear cases.

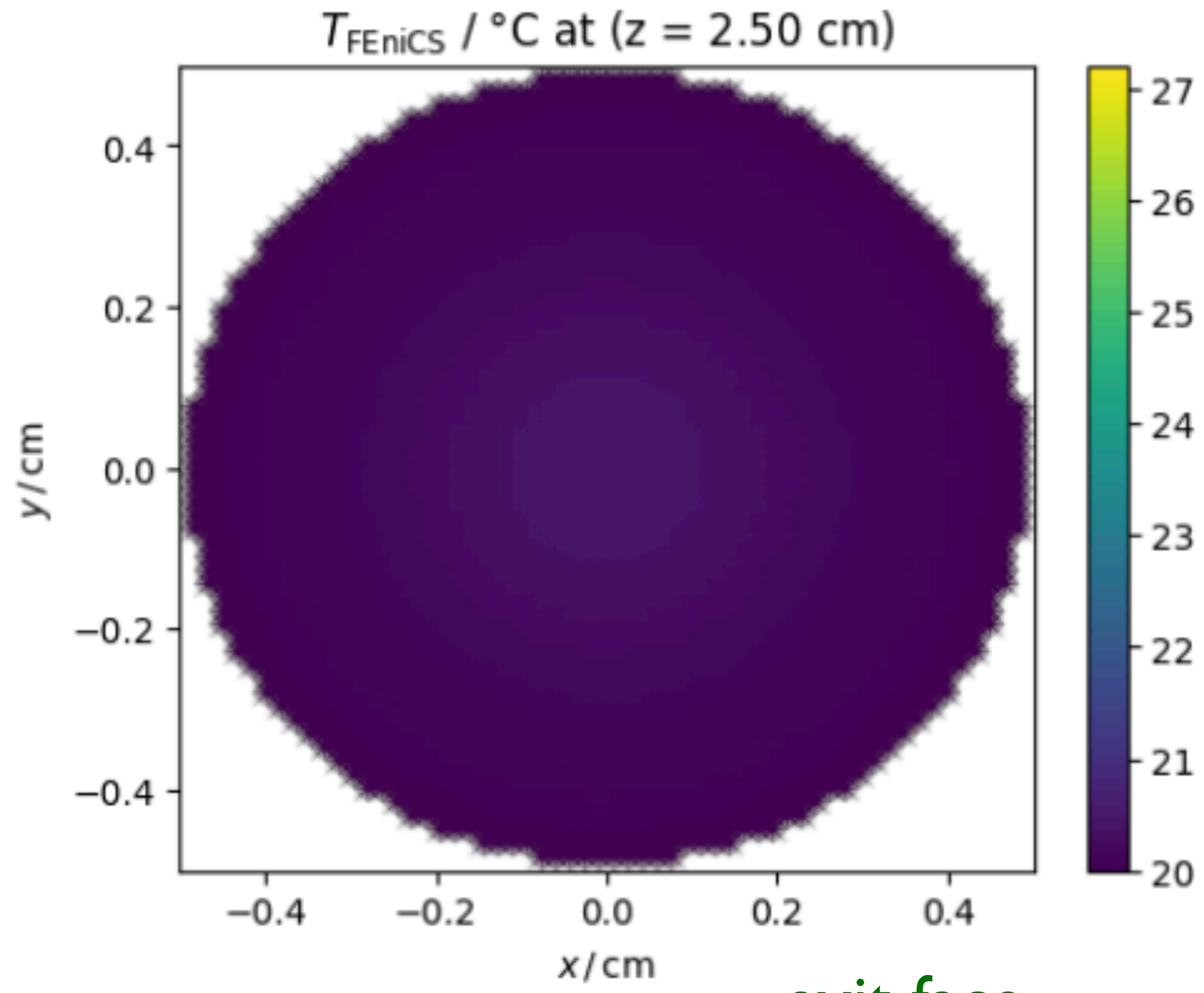
entrance face











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Lessons Learned, and Next Steps:

Lessons Learned:

- Obtain thermal data characteristic of *your* crystal.
- At room temperature and present power levels, the linear heat equation appears adequate.

But ... know / determine your thermal diffusivity, as well as other thermodynamic data.

Next Steps:

- Explore effects at cryogenic temperatures, where α varies more rapidly with temperature.
- Include anisotropic heat conduction — *i.e.* non-axisymmetric (astigmatic) thermal lensing.
- Investigate thermal expansion — *i.e.* additional lensing from thermally induced curvature.

See also the following Jupyter notebook, which you can use on RadiaSoft's Jupyter server:

https://github.com/radiasoft/rslaser/blob/master/examples/thermal/NLThermalSaturation.ipynb

Thank you!

This work is supported by the US Department of Energy, Office of Science, Office of High Energy Physics under Award Numbers DE-SC0020931 and DE-AC02-05CH11231.

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