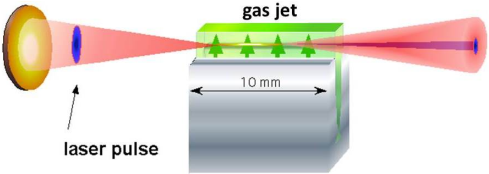
A preliminary analysis for efficient laser wakefield acceleration

Gaetano Fiore, Universitá "Federico II", and INFN, Napoli

20th Advanced Accelerator Concepts Workshop
Hyatt Regency Long Island, NY, November 7-11, 2022

Introduction

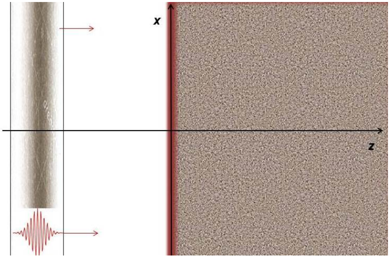
Laser Wake-Field Acceleration (WFA) [Tajima, Dawson 79] is the first and prototypical mechanism of extreme acceleration of charged particles along short distances: electrons "surf" a plasma wave (PW) driven by a very short laser pulse, e.g. in a supersonic diluted gas jet.



Dynamics is ruled by Maxwell equations coupled to a kinetic theory for plasma electrons, ions. Today these eqs can be solved via more and more powerful PIC codes. However such computations involve huge costs for each choice of the input data. Therefore it is crucial to run them after a preliminary selection of the input parameters based on a simpler model.

Here we present conditions enabling a hydrodynamic description (HD) of the **impact of a very short and intense laser pulse onto a cold diluted plasma at rest** as long as possible, study the induced PW and its wave-breaking (WB) at density inhomogeneities [Dawson 59], derive preliminary conditions for optimizing *self-injection* of small bunches of electrons in the PW and their LWFA.

We assume x, y-independence of: $\widetilde{n}_0$ = initial plasma density; the pulse = transverse *plane* electromagnetic (EM) traveling-wave in z-direction. We argue that with a "real" pulse, i.e. with a spot size $R < \infty$ (not too small), the results will hold inside the causal cone trailing the pulse. Up to shortly after the impact we can regard the pulse as



undepleted, ions as immobile. This allows to describe the remaining dynamics by a family (parametrized by $Z > 0$) [GF14-18] of decoupled non-autonomous and highly nonlinear Hamilton equations for systems with 1 degree of freedom; $\xi = ct - z$ instead of time t as an independent variable. After the laser-plasma interaction the Jacobian $\hat{J}$ of the map from Lagrangian to Eulerian coordinates is *linear-quasiperiodic* (LQP) in ξ with period $\xi_H(Z) \equiv ct_H(Z)$, i.e. of the form

$$\hat{J}(\xi, Z) = a(\xi, Z) + \xi\, b(\xi, Z); \tag{1}$$

a, b are ξ_H-periodic in ξ, and b has zero mean over a period, cf [Brantov... 08]

Plan

Setup & Plane model

$\mathbf{v}_e(0,\mathbf{x})=\mathbf{0}$. Input = nontrivial initial data:

1. the function $\widetilde{n_0}(z) \geq 0$, with $\widetilde{n_0}(z)=0$ if $z<0$, $\widetilde{n_0}(z) \leq n_b \in \mathbb{R}^+$ if $z>0$, yielding the initial electron and proton densities n_e, n_p:

$$n_e(0,\mathbf{x})=n_p(0,\mathbf{x})=\widetilde{n_0}(z); \tag{2}$$

2. the vector-valued function $\boldsymbol{\epsilon}^\perp(\xi)$ yielding the initial laser-pulse EM fields:

$$\mathbf{E}(t,\mathbf{x}) = \mathbf{E}^\perp(t,\mathbf{x}) = \boldsymbol{\epsilon}^\perp(ct-z), \qquad \mathbf{B} = \mathbf{B}^\perp = \mathbf{k}\times\mathbf{E}^\perp \qquad \text{if } t \leq 0, \tag{3}$$

with support$(\boldsymbol{\epsilon}^\perp) \subseteq [0, l]$ fulfilling $l \lesssim \sqrt{\pi mc^2/n_b e^2}$, or more precisely (14a): the pulse reaches the plasma at $t=0$ and overcomes all e^- before their z reach the first negative minimum (*essentially short* pulse).

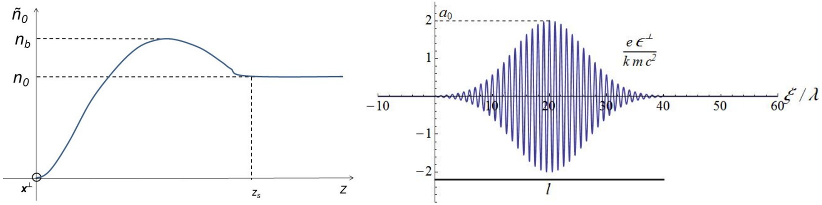
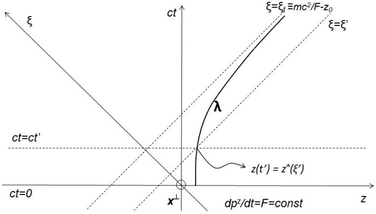
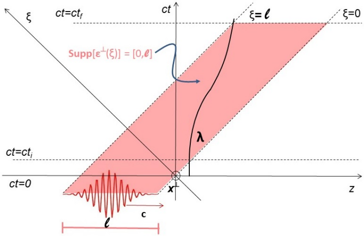


Figure 1: In particular, interested in $\widetilde{n_0}(z)$ with a downramp+plateau

As no particle can travel at speed c, $\tilde{\xi}(t)=ct-z(t)$ is strictly growing: we can adopt $\xi=ct-z$ as the independent parameter on the worldline λ & in EoM.





$$\text{initial position}\quad \mathbf{X}=(X,Y,Z)\ \underset{\substack{\mathbf{X}_e(t,\cdot)\\ \hat{\mathbf{X}}_e(\xi,\cdot)}}{\overset{\substack{\mathbf{x}_e(t,\cdot)\\ \hat{\mathbf{x}}_e(\xi,\cdot)}}{\rightleftarrows}}\ \mathbf{x}=(x,y,z)\quad \text{position at } t,\ \xi \quad \text{must be 1-to-1}$$

$$\mathbf{X}_e(t,\mathbf{x})=\hat{\mathbf{X}}_e(ct-z,\mathbf{x})$$

Eulerian $f(t,\mathbf{x})=\tilde{f}(t,\mathbf{X})=\hat{f}(\xi,\mathbf{X})$ Lagrangian observables. Use CGS units.

Dimensionless: $\boldsymbol{\beta}\equiv\frac{\dot{\mathbf{x}}}{c}$, $\gamma\equiv\frac{1}{\sqrt{1-\boldsymbol{\beta}^2}}$, 4-vel. $u=(u^0,\mathbf{u})\equiv(\gamma,\gamma\boldsymbol{\beta})=\left(\frac{p^0}{mc^2},\frac{\mathbf{p}}{mc}\right)$

$s\equiv\gamma-u^z$. $s\to 0$ implies $u^z\to\infty$.

Reduction of the dynamics to decoupled ODEs...

PDEs: **Lorentz-Maxwell & continuity eq. for the electron fluid**; + in. cond.
Are reduced to the family (parametrized by Z) of ordinary Cauchy problems

$$\hat{\Delta}' = \frac{1+v}{2\hat{s}^2} - \frac{1}{2}, \qquad \hat{s}' = K\left\{\widetilde{N}\left[Z+\hat{\Delta}\right] - \widetilde{N}(Z)\right\}, \tag{4}$$

$$\hat{\Delta}(0,Z) = 0, \qquad \hat{s}(0,Z) = 1 \tag{5}$$

[GF18] ($\hat{f}' \equiv \partial\hat{f}/\partial\xi$) in the unknowns $\hat{\Delta}(\xi,Z) \equiv \hat{z}_e(\xi,Z) - Z$, $\hat{s}(\xi,Z)$, in the spacetime region where $\hat{\mathbf{x}}_e(\xi,\cdot) : \mathbf{X} \mapsto \mathbf{x}$ is one-to-one and the pulse is not significantly modified by its interaction with the plasma. Here $K := \frac{4\pi e^2}{mc^2}$, and

$$v(\xi) := \left[\frac{e\boldsymbol{\alpha}^{\perp}(\xi)}{mc^2}\right]^2, \qquad \boldsymbol{\alpha}^{\perp}(\xi) := -\int_{-\infty}^{\xi} d\zeta\, \boldsymbol{\epsilon}^{\perp}(\zeta), \tag{6}$$

$$\widetilde{N}(Z) := \int_0^Z d\zeta\, \widetilde{n}_0(\zeta), \qquad \mathcal{U}(\Delta;Z) := K\int_0^{\Delta} d\zeta\,(\Delta-\zeta)\,\widetilde{n}_0(Z+\zeta)\,. \tag{7}$$

Clearly $v \geq 0$, and $\widetilde{N}(Z)$ grows with Z.

...which are Hamiltonian for 1-dim systems

For each $Z \geq 0$ (4) are **Hamilton equations** $q' = \partial\hat{H}/\partial p$, $p' = -\partial\hat{H}/\partial q$ of a **1-dim system**: $\xi, \hat{\Delta}, -\hat{s}$ play the role of t, q, p, and the Hamiltonian reads

$$\hat{H}(\hat{\Delta}, \hat{s}, \xi; Z) := \frac{\hat{s}^2 + 1 + v(\xi)}{2\hat{s}} + \mathcal{U}(\hat{\Delta}; Z) \tag{8}$$

up to mc^2. For $\xi > l$ v=const, $\hat{H}=h$=const, (4) are autonomous and can be solved by quadrature; if $Z>0$ the solutions are periodic in ξ with period $\xi_H(Z)$.

All other unknowns expressible in terms of $(\hat{\Delta}, \hat{s})$:

$$\frac{\hat{\mathbf{p}}^\perp}{mc} = \hat{\mathbf{u}}^\perp = \frac{e\,\boldsymbol{\alpha}^\perp(\xi)}{mc^2}, \qquad \frac{\hat{p}^z}{mc} = \hat{u}^z = \frac{1+\hat{\mathbf{u}}^{\perp 2} - \hat{s}^2}{2\hat{s}}, \qquad \frac{\hat{p}^0}{mc^2} = \hat{\gamma} = \frac{1+\hat{\mathbf{u}}^{\perp 2} + \hat{s}^2}{2\hat{s}}, \tag{9}$$

$$\hat{\mathbf{x}}_e^\perp(\xi, \mathbf{X}) - \mathbf{X}^\perp = \int_0^\xi d\eta\, \frac{\hat{\mathbf{u}}^\perp(\eta)}{\hat{s}(\eta, Z)}, \qquad \hat{z}_e(\xi, \mathbf{X}) - Z = \hat{\Delta}(\xi, Z). \tag{10}$$

As $\boldsymbol{\alpha}^\perp(\xi)$ is independent of $\mathbf{X}$ so are $\hat{\mathbf{p}}^\perp, \hat{\mathbf{u}}^\perp$; as $\hat{s}, \hat{\Delta}$ are independent of X, Y so are $\hat{p}^z, \hat{u}^z, \Delta\hat{\mathbf{x}}_e$. Replacing $(\xi, \mathbf{X}) \mapsto \left(ct - z, \hat{\mathbf{X}}_e(ct - z, \mathbf{x})\right)$ in the arguments we get their Eulerian counterparts, e.g. $n_e(t, z) = \left[\frac{\hat{\gamma}\,\widetilde{n_0}}{\hat{s}\,\hat{J}}\right]_{(\xi, Z) = \left(ct - z, \hat{Z}_e(ct - z, z)\right)}$.

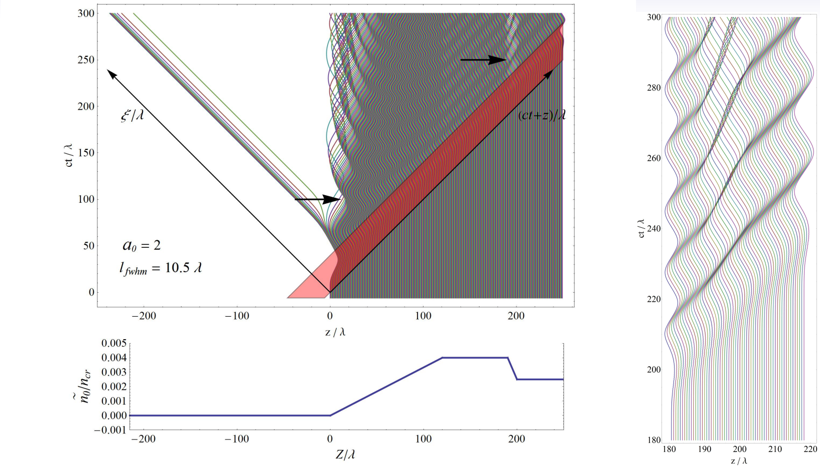


Figure 2: e^- WL induced by the pulse of fig. 1 [Brantov et al 08] on the initial density $\widetilde{n}_0(Z)$ plotted below: WL of $Z \simeq 0$ e^- stray left away (*slingshot effect*), WL of other up-ramp e^- first cross after $\sim 5/4$ oscillations (left arrow), WL of down-ramp e^- first cross after $\sim 7/4$ oscills (right arrow + zoom)

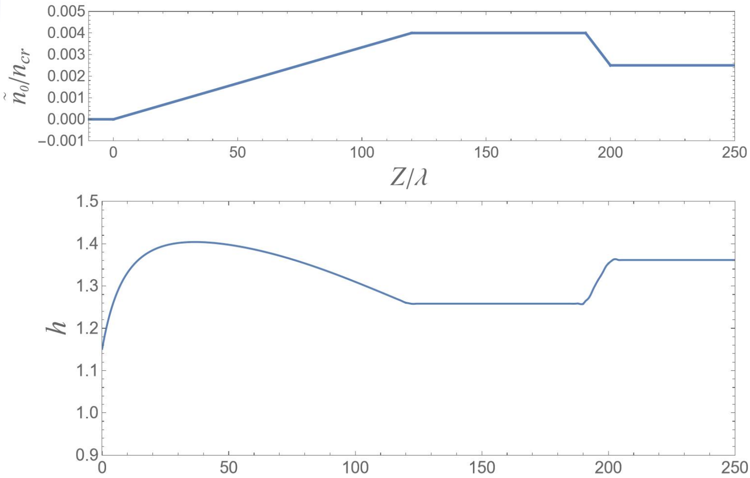


Figure 3: Up: normalized initial electron density with a downramp and a final plateau. Down: corresponding normalized final energy $h(Z)$ of the Z-electrons after interacting with the pulse described in fig. 1. These conditions are as in section III.B of Ref. [Brantov et al, PoP2008].

Auxiliary problem: constant initial density

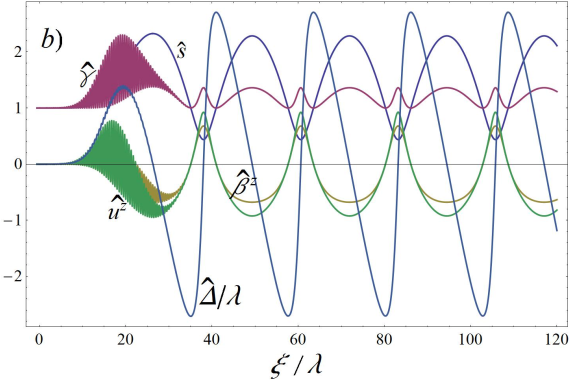
If $\widetilde{n}_0(Z) \equiv n_0 =$ **const**, then (4) and its solution are in fact Z-independent:

$$\Delta' = \frac{1+v}{2s^2} - \frac{1}{2}, \qquad s' = M\Delta, \qquad \Delta(0)=0, \quad s(0)=1, \; (11)$$

where $M \equiv Kn_0$, $\mathcal{U}(\Delta, Z) \equiv M\Delta^2/2$: relativistic harmonic oscillator.

Right: Solution of (11) if the pulse is as in fig. 1, i.e. with length $l = 40\lambda$, linear polarization, peak amplitude $a_0 \equiv \lambda e E_M^{\perp}/2\pi mc^2 = 2$ (if $\lambda = 0.8\mu$m this leads to a peak intensity $I = 1.7 \times 10^{19}$W/cm^2), and $n_0 \equiv n_c/400$, where $n_c \equiv mc^2/e^2\lambda^2$. As a result, $E = 1.4mc^2$.

Note: $\hat{s}$ is insensitive to fast oscillations of $\epsilon^{\perp}$!



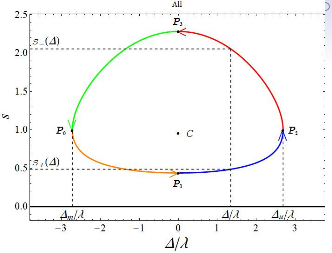


Figure 4: Corresponding phase portrait (at $\xi > l$) .

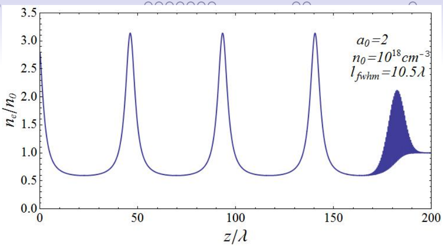


Figure 5: Normalized electron density as a function of z at $ct = 120\lambda$.

The Z-independent period $\bar{\xi}_H = c\,\bar{t}_H$ is

$$\bar{\xi}_H(n_0, h) = 4\sqrt{\frac{2(h+\gamma^\perp)}{Kn_0}}\left[\mathcal{E}(\alpha) - \frac{\gamma^\perp}{h+\gamma^\perp}\mathcal{K}(\alpha)\right], \qquad \alpha := \sqrt{\frac{h-\gamma^\perp}{h+\gamma^\perp}}; \tag{12}$$

$\mathcal{K}, \mathcal{E}$ are the complete elliptic integrals of the 1st, 2nd kind. (12) reduces to $ct_H^{nr} \equiv \sqrt{\frac{\pi mc^2}{n_0 e^2}}$ in the nonrelativistic limit $h \to 1$, $\gamma^\perp \to 1$, to $ct_H^{ur} \simeq \frac{15\pi}{8}\sqrt{\frac{2h}{M}}$ in the ultrarelativistic limit $h \to \infty$. $\hat{J} \equiv \frac{\partial \hat{z}_e}{\partial Z} = 1$, $\hat{Z}_e(\xi, z) = z - \Delta(\xi)$, and $n(t,z), \mathbf{u}(t,z), \ldots$ evolve as travelling waves:

$$n_e(t,z) = \frac{n_0}{2}\left[1 + \frac{1+v(ct-z)}{s^2(ct-z)}\right] = \frac{n_0}{1-\beta^z(ct-z)}. \tag{13}$$

Apriori estimates of $\hat{\Delta}, \hat{s}$ *for* $0 \le \xi \le l$

By (4) $\hat{\Delta}$, $\hat{s}-1$ grow positive for small $\xi > 0$. $\hat{\Delta}(\xi,Z)$ reaches a maximum at $\tilde{\xi}_1(Z) \equiv$ the smallest $\xi > 0$ such that $1+v(\xi) = \hat{s}^2(\xi,Z)$. $\hat{s}(\xi,Z)$ grows as long as $\hat{\Delta}(\xi,Z) \ge 0$, reaches a maximum at the first zero $\tilde{\xi}_2 > \tilde{\xi}_1$ of $\hat{\Delta}(\xi,Z)$ decreases for $\xi > \tilde{\xi}_2$, while $\hat{\Delta}(\xi,Z) < 0$. Let $\tilde{\xi}_3(Z) \equiv$ the smallest $\xi > \tilde{\xi}_2$ s.t. $\hat{s}(\xi,Z) = 1$.

$$\begin{array}{llll} \text{The pulse is } \textit{essentially short if} & l < \tilde{\xi}_3(Z) & \Leftrightarrow & \hat{s}(\xi, Z) \ge 1, \\ \text{the pulse is } \textit{strictly short if} & l < \tilde{\xi}_2(Z) & \Leftrightarrow & \hat{\Delta}(\xi, Z) \ge 0 \end{array} \qquad (14)$$

for all $xi \in [0, l]$, $Z \ge 0$. Essential shortness is compatible with maximum pulse-to-plasma energy transfer, which takes place at a suitable $l \sim \tilde{\xi}_2$.

Let $\Delta^{(0)}(\xi) \equiv \int_0^\xi d\eta \, \frac{v(\eta)}{2}$, $\Delta_u \equiv \Delta^{(0)}(l)$, $n_u, n_d > 0$ be some local bounds for $\widetilde{n_0}$,

$$n_d(Z) \le \widetilde{n_0}(z) \le n_u(Z) \qquad \forall z \in \left[Z, Z+\Delta_u\right], \qquad (15)$$

$\Delta_d(Z)$ be the negative solution of the eq. $\mathcal{U}(\Delta;Z) = Kn_u(Z)\Delta_u^2/2$, $n_u'(Z) \equiv \max_{z\in[Z+\Delta_d,Z]} \{\widetilde{n_0}(z)\}$. We abbreviate $M_u \equiv Kn_u$, $M_d \equiv Kn_d$, $M_u' \equiv Kn_u'$,

$$s_u \equiv 1+\frac{M_u}{2}\Delta_u^2+\sqrt{\left(1+\frac{M_u}{2}\Delta_u^2\right)^2-1}, \qquad g(\xi,Z) \equiv \frac{M_u(Z)}{2}\int_0^\xi d\eta \, (\xi-\eta)\, v(\eta), \qquad (16)$$

$$\hat{s}^{(1)}(\xi,Z) \equiv \min\left\{s_u(Z), 1+g(\xi,Z)\right\}, \quad f(\xi,Z) \equiv \int_0^{\xi} d\eta\,(\xi-\eta)\left\{\frac{1+v(\eta)}{\left[\hat{s}^{(1)}(\eta,Z)\right]^2}-1\right\}$$

$$\Delta^{(1)}(\xi,Z) \equiv \max\left\{\Delta_d, f'(\xi,Z)\right\},$$

$$\hat{s}^{(2)}(\xi,Z) \equiv \begin{cases} 1+\frac{M_d}{2}f(\xi,Z) & 0 \le \xi \le \tilde{\xi}_2^{(1)} \\ \max\left\{\hat{s}_d, 1+\left[\frac{M_d}{2}-\frac{M_u'}{2}\right] f\left(\tilde{\xi}_2^{(1)},Z\right)+\frac{M_u'}{2}f(\xi,Z)\right\} & \tilde{\xi}_2^{(1)} < \xi \le \tilde{\xi}_3'. \end{cases}$$

where $\tilde{\xi}_2^{(1)}(Z) < \tilde{\xi}_2$ is the maximum point of $f(\xi,Z)$ and $\tilde{\xi}_3' := \min\{l, \tilde{\xi}_3\}$.
Some sufficient conditions for the pulse to be resp. strictly, essentially short:

$$f'(l,Z) \ge 0 \quad \Rightarrow \quad \tilde{\xi}_2(Z) > l, \tag{18}$$

$$f(l,Z) \ge \left(1-\frac{n_d}{n_u'}\right)\max_{\xi} f(\xi,Z) \quad \Rightarrow \quad \tilde{\xi}_3(Z) > l. \tag{19}$$

Proposition 1. If the pulse is *essentially short*, then for all $\xi \in [0,l]$

$$\begin{aligned} &\Delta_u \ge \Delta^{(0)}(\xi,Z) \ge \hat{\Delta}(\xi,Z) \ge \Delta^{(1)}(\xi,Z) \ge \Delta_d, \\ &s_u \ge \hat{s}^{(1)}(\xi,Z) \ge \hat{s}(\xi,Z) \ge \hat{s}^{(2)}(\xi,Z) \ge 1. \end{aligned} \tag{20}$$

Hydrodynamic regime up to wave-breaking (WB)

Map $\hat{\mathbf{x}}_e(\xi,\cdot):\mathbf{X}\mapsto\mathbf{x}$ invertible, and hydrodynamic regime justified, as long as

$$\hat{J}\equiv\left|\frac{\partial\hat{\mathbf{x}}_e}{\partial\mathbf{X}}\right|=\frac{\partial\hat{z}_e}{\partial Z}=1+\varepsilon>0,\qquad\qquad \varepsilon\equiv\frac{\partial\hat{\Delta}}{\partial Z}. \tag{21}$$

$\hat{J}(\xi,Z)\leq 0$: $\exists Z'\neq Z$, s.t. $\hat{z}_e(\xi,Z')=\hat{z}_e(\xi,Z)$, i.e. Z',Z electrons collide, $\exists$ WB.

$$n_e(t,z)=\left[\frac{\hat{\gamma}\,\widetilde{n_0}}{\hat{s}\,\hat{J}}\right]_{(\xi,Z)=\left(ct-z,\hat{Z}_e(ct-z,z)\right)}\quad \text{diverges where } \hat{J}=0. \tag{22}$$

Proposition 2. If $\xi>l$ then $\hat{J}$ is LQP in ξ, because

$$\hat{J}(\xi+n\xi_H,Z)=\hat{J}(\xi,Z)-n\frac{\partial\xi_H}{\partial Z}\Delta'(\xi,Z),\qquad \forall n\in\mathbb{N},\quad Z\geq 0. \tag{23}$$

Proof: Differentiate the identity $\Delta[\xi+n\xi_H(Z),Z]=\Delta(\xi,Z)$ with respect to Z, use $\Delta'[\xi+n\xi_H,Z]=\Delta'[\xi,Z]$. $\square$

By (23) we can extend our knowledge of $\hat{J}$ from $[l,l+\xi_H[$ to all $\xi\geq l$.
(23) agrees with (1) if $b(\xi,Z)\equiv-\hat{\Delta}'(\xi,Z)\,\frac{\partial\log\xi_H}{\partial Z}$, $a(\xi,Z)\equiv\hat{J}(\xi,Z)-\xi b(\xi,Z)$.

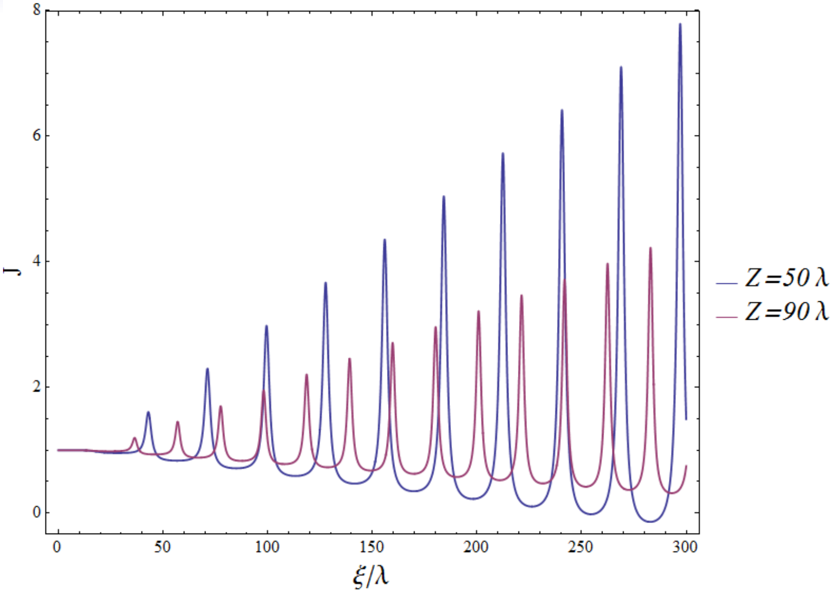


Figure 6: $\hat{J}(\xi,Z) = 1+\varepsilon(\xi,Z)$ for $Z = 50\lambda, 90\lambda$ in the situation of fig. 2.

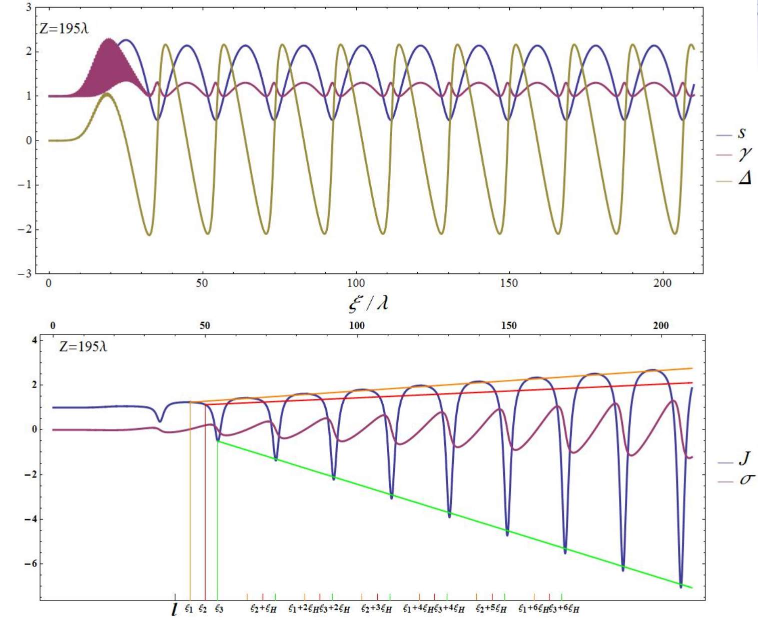


Figure 7: Evolution of $\hat{s}, \hat{\Delta}, \hat{\gamma}$ (up) and $\hat{J}, \hat{\sigma}$ (down), in the situation of fig. 2, for the $Z = 195\lambda$ electron layer; there $\widetilde{n_0}(Z)$ decreases, $\Phi(Z) > 0$, $\vartheta(Z)$...

Let $\sigma \equiv l\,\partial\hat{s}/\partial Z$, $\kappa \equiv (1+v)/l\hat{s}^3$, $\check{n}(\xi,Z) \equiv \widetilde{n_0}\,[\hat{z}_e(\xi,Z)]$,

$$\boldsymbol{\chi} \equiv \begin{pmatrix} \varepsilon \\ \sigma \end{pmatrix}, \qquad A \equiv \begin{pmatrix} 0 & -\kappa \\ Kl\check{n} & 0 \end{pmatrix}, \qquad \boldsymbol{\lambda} \equiv \begin{pmatrix} 0 \\ Kl[\check{n}-\widetilde{n_0}] \end{pmatrix}.$$

Differentiating (4) with respect to Z we find that $\boldsymbol{\chi}$ fulfills the Cauchy problem

$$\boldsymbol{\chi}' = A\boldsymbol{\chi} + \boldsymbol{\lambda}, \qquad \boldsymbol{\chi}(0) = \begin{pmatrix} 0 \\ 0 \end{pmatrix}. \tag{24}$$

The solution of (24) can be expressed as

$$\boldsymbol{\chi}(\xi) = G(\xi)\int_0^{\xi} d\eta\; G^{-1}(\eta)\boldsymbol{\lambda}(\eta), \tag{25}$$

where G is a 2×2-matrix solving $G' = AG$, $\det G(0) \neq 0$.

For $\xi \geq l$ the eqs $G' = AG$ and (24) are ξ_H-periodic; LQP follows also from Floquet Theorem.

Bounds on the Jacobian for small $\xi>0$, and no WBDLPI conditions

To bound ε,σ for small ξ we introduce the Liapunov function

$$V\equiv\varepsilon^2+b\,\sigma^2,\qquad b\equiv 1/M_u l^2. \tag{26}$$

$|\varepsilon|\le\sqrt{V}$, $V(0,Z)=0$. Using (24) and the Comparison Principle one shows

$$\begin{aligned}
|\varepsilon(\xi,Z)| &\le \delta\sqrt{M_u}\int_0^l d\eta\,\exp\left\{\frac{\sqrt{M_u}}{2}\left[(l-\eta)\,\delta+\int_\eta^l d\zeta\,\mathcal{D}(\zeta)\right]\right\}=:Q_2\\
&\le \delta\sqrt{M_u}\int_0^l d\eta\,\exp\left\{\frac{\sqrt{M_u}}{2}\left[(l-\eta)\,\delta+\int_\eta^l d\zeta\,\tilde v(\zeta)\right]\right\}=:Q_1\\
&\le \frac{2\delta}{\tilde v_M+\delta}\left\{\exp\left[\frac{\tilde v_M+\delta}{2}\sqrt{M_u}\,l\right]-1\right\}=:Q_0\qquad\forall\xi\in[0,\tilde\xi_3'],\\
\text{where}\quad \mathcal{D}(\xi,Z) &:= \max\left\{\frac{1+v(\xi)}{[\hat s^{(2)}(\xi,Z)]^3}-1\,,\;1-\frac{1+v(\xi)}{[\hat s^{(1)}(\xi,Z)]^3}\right\}\\
&\le \tilde v(\xi):=\max\{v(\xi),1\}\le\max\{v_M,1\}=:\tilde v_M,
\end{aligned}$$

Proposition 4. If (19) and either $Q_0(Z)<1$, or $Q_1(Z)<1$, or $Q_2(Z)<1$ are fulfilled, then no *Wave-breaking during the laser-plasma interaction* (WBDLPI) involves the Z-electrons. If this occurs for all Z, then no WBDLPI anywhere.

Nonrelativistic regime (NR): $v \ll 1$, $\hat{s} \simeq 1$, $l\kappa \simeq 1$, $\Delta_u \ll l$, $\mathcal{D} \simeq 0$; $Q_2 < 1$ boils down to

$$r(Z) := \delta(Z)\sqrt{Kn_u(Z)}\,l \;<\; 0.81, \qquad \delta(Z) := 1 - \frac{n_d(Z)}{n_u(Z)}. \tag{27}$$

(27) is automatically satisfied if $\sqrt{Kn_u(Z)}l < 0.81$, as $\delta \leq 1$ by def., otherwise is a very mild condition on the relative variation δ of $\widetilde{n_0}(z)$ across $[Z, Z{+}\Delta_u]$. If in some interval $]0, \bar{Z}{+}\Delta_u]$ $\widetilde{n_0}(Z)$ is C^1 (at least piecewise), and

$$0 \leq \frac{d\sqrt{\widetilde{n_0}}}{dZ} \leq \frac{0.81}{2l\,\Delta_u\sqrt{K}} = \frac{2\times 10^5}{l\,\Delta_u}\mathrm{cm}^{-1/2}, \tag{28}$$

then it fulfills (27): no WBDLPI! Qualitatively the same also in relativistic regime. Applies to most cases of physical interest:

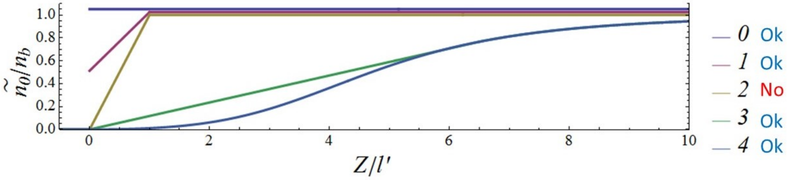


Figure 8: If target = supersonic gas jet then $\widetilde{n_0}(z)$ is of type 4; if target = aerogel all types of $\widetilde{n_0}(z)$ are possible.

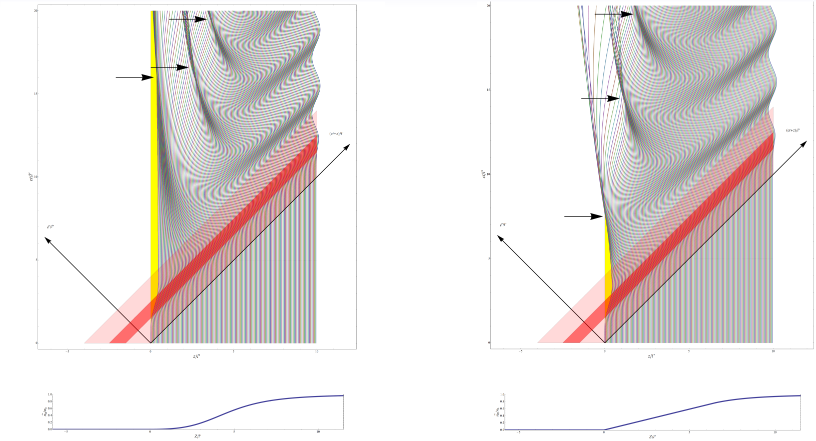


Figure 9: Down: normalized $\widetilde{n}_0 \in C^1$ (left), $\widetilde{n}_0 \in C^0$ (right). Up: WL of the associated Z-electrons interacting with the pulse of fig. 1. Arrows pinpoint intersections of WL.

Motion of a test electron in the plasma wave

The $\hat{z}_i, \hat{s}_i$ of a test electron injected at $\xi = \xi_0 > l$ in $\hat{z}_i(\xi_0) = z_0 > z_s$ (PW behind the pulse along the plateau) with $\hat{s}_i(\xi_0) = s_0$ evolve as follows:

$$\hat{s}_i(\xi) = s_0 + M\int_{\xi_0}^{\xi} dy\, \Delta(y) = \delta s + s(\xi), \quad \hat{z}_i(\xi) = z_0 + \int_{\xi_0}^{\xi} \frac{dy}{2}\left[\frac{\gamma_i^{\perp\, 2}}{\hat{s}_i^2(y)} - 1\right] \tag{29}$$

($\gamma_i^{\perp\, 2}$=const, tipically $\simeq 1$). Note: $\hat{s}_i(\xi) - s(\xi) = \delta s \equiv s_0 - s(\xi_0) =$ const!
If $s_i^m \equiv s_m + \delta s \leq 0$, then $\hat{s}_i(\xi)$ vanishes at some $\xi_f > \xi_0$ (while $\hat{s}, s$ can vanish nowhere!): since $t_f = \infty$, this yields **an electron trapped in the PW**!
If $s_i^m < 0$, for $\xi \simeq \xi_f$ we have $\hat{s}_i(\xi) \simeq |s'(\xi_f)|\,(\xi_f - \xi) = M\,|\Delta(\xi_f)|\,(\xi_f - \xi)$, whence

$$\hat{z}_i(\xi) \simeq \frac{\gamma_i^{\perp\, 2}}{2\,[M\Delta(\xi_f)]^2\,(\xi_f - \xi)} \xrightarrow{\xi \to \xi_f} \infty. \tag{30}$$

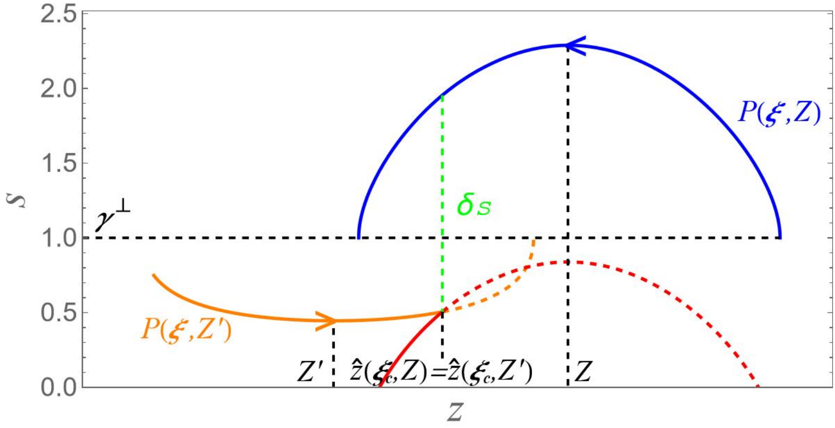
Solving (30) for $\xi_f - \xi$, we can express $\hat{s}_i, \hat{\gamma}_i$ as functions of z_i, and find

$$\gamma_i = \frac{\gamma_i^{\perp\, 2}}{2s_i} + \frac{s_i}{2} \simeq |M\Delta(\xi_f)|\; z_i \xrightarrow{z_i \to \infty} \infty; \tag{31}$$

in this simplified model trapped test electrons *cannot dephase* (as the pulse travels faster, at c) and **their energy grows $\propto$ the travelled distance**.

Self-Injection by WB

If for some $Z' \leq z_s \leq Z$ (with $Z - Z' < \Delta_M(Z') - \Delta_m$) the Z'-electron layer (moving forward) first hits the Z-electron layer (moving backward) at some $\xi = \xi_c > l$, i.e. $\hat{z}(\xi_c, Z) = \hat{z}(\xi_c, Z') \equiv z_0$, few Z'-electron will be injected in the PW at $z = z_0$ without changing $\hat{s}$, and (29-31) apply with $s_0 = \hat{s}(\xi_c, Z')$.



We have computed the range of energies that self-injected e^- arising from gentle WB collisions as in the right part of fig. 2: the maximum energy quantitatively agrees with the results of PIC simulations in [Brantov et al 08].

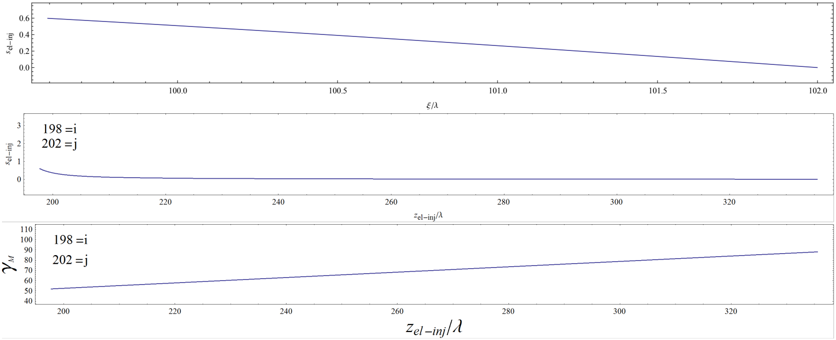


Figure 10: Up: s_i as a function of ξ for the highest-energy electrons (HEE), which are self-injected by the collision of the $Z = 198\lambda$ electrons with the $Z = 202\lambda$ ones ($\xi_c = 99\lambda$). Center: The path in the $z_i - s_i$ plane of the same HEE. Down: The Lorentz factor as a function of z_i of the same HEE.

For fixed z_i, n_0, γ_i is maximal if $-\delta s = \gamma^\perp \simeq 1$, so that $\Delta(\xi_f)| = \Delta_m$:

$$\gamma_i^M(z_i, n_0) \simeq M\,|\Delta_m|\,z_i = \sqrt{2K\,f(n_0)}\,z_i, \qquad f(n_0) \equiv n_0\left[\bar{h}(n_0) - \gamma^\perp\right]. \quad (32)$$

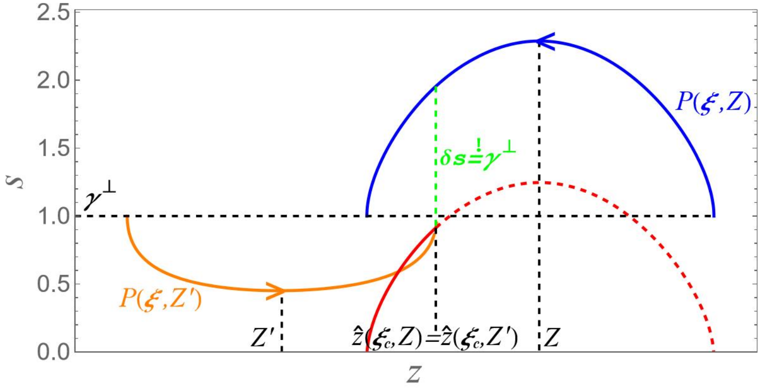
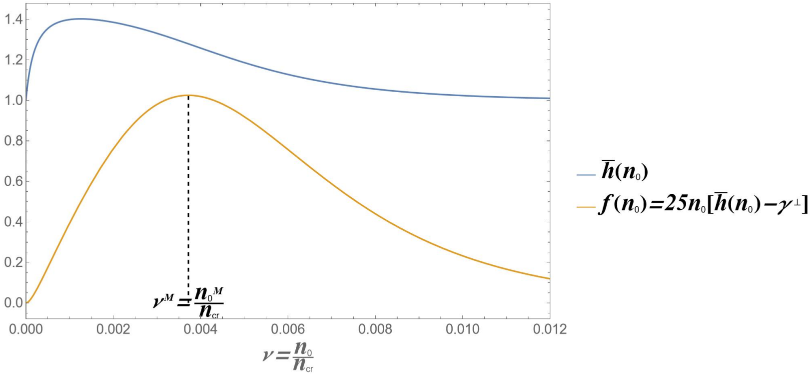


Figure 11

1st Step: We choose $n_0 = n_0^M \equiv$ max point of f to maximize γ_i^M for fixed z_i:



We get optimal self-injection if we can find small intervals of Z,Z' & $\xi_c(Z,Z')$ s. t.

$$\hat{z}(\xi_c, Z) = \hat{z}(\xi_c, Z'), \qquad s(\xi_c) - \hat{s}(\xi_c, Z') = \gamma^\perp; \tag{33}$$

then $\hat{\gamma}^M(\hat{z}_i)$ is approximately as in (32).

The **2nd step** in our optimization process is the determination of pairs Z', Z fulfilling (33). In (33) also $\xi_c > l$ is an unknown; but since for $\xi > l$ the phase portraits are determined, we can first get rid of ξ_c by expressing the $\hat{s}$'s as functions of $\hat{\Delta}$'s; this transforms (33) into

$$\Delta' - \Delta = Z - Z', \quad s_{+}(\Delta, Z) - s_{-}(\Delta', Z') = \gamma^{\perp}. \tag{34}$$

constrained by $\Delta' \in [0, \Delta_M(Z')], \quad \Delta \in [\Delta_m(Z), 0]$. Clearly solutions can only exist if

$$Z - Z' < \Delta_M(Z') - \Delta_m(Z), \qquad s_M(Z) - s_{-}(Z') > \gamma^{\perp}. \tag{35}$$

Solutions of (34) can be graphically visualized by families of figures like fig. 11. We determine the order of magnitude of $Z - Z'$ as $1/2$ rhs(35a):

$$Z - Z' \sim \frac{\Delta_M(Z') - \Delta_m(Z)}{2} \simeq \frac{\sqrt{2f(n_0')}}{\sqrt{K} n_0'} + \frac{\sqrt{2f(n_0^M)}}{\sqrt{K} n_0^M} \simeq \frac{\sqrt{2f(n_0^M)}}{\sqrt{K} n_0^M} \left(1 + \frac{n_0^M}{n_0'}\right), \tag{36}$$

where we have abbreviated $n_0' \equiv \widetilde{n}_0(Z')$ and used that $f(n_0') \simeq f(n_0^M)$ because $f'(n_0^M) = 0$

Next we look for the unknown ξ_c.

A s-jump as in fig. 11 can occur at $\xi=\xi_c(Z,Z')$ only if, for some $i,j\in\mathbb{N}$, with $j>i$, $P(\xi,Z)$ has completed $i+\frac{1}{2}$ orbits and $P(\xi_c,Z)$ is on its arc $\overset{\frown}{P_3P_0}$ (see fig. 4), while $P(\xi,Z')$ has completed j orbits and $P(\xi_c,Z')$ is on its arc $\overset{\frown}{P_1P_2}$:

$$j\,\xi_H(Z')\;\sim\;\xi_c(Z,Z')\;\sim\;\left(i+\frac{1}{2}\right)\xi_H(Z)$$

or approximately $\bar{\xi}_H(n_0',h')\simeq\xi_H(Z')\sim\frac{2i+1}{2j}\xi_H(Z)=\frac{2i+1}{2j}\bar{\xi}_H\left[n_0^M,\bar{h}(n_0^M)\right]$; via (12) this allows a first rough determination of n_0' as

$$n_0'\;\simeq\;\frac{(2i+1)^2}{(2j)^2}\,n_0^M. \tag{37}$$

The smallest i,j consistent with no WBDLPI are $i=1,j=2$, whereby $n_0'=\frac{16}{9}n_0^M$.

By more complicated calculations one finds a more accurate relation yielding n_0' in terms of n_0^M. Imposing $\widetilde{n}_0(Z')=n_0'$, and thus determining via (36) also the best slope of $\widetilde{n}_0(Z)$ before the plateau, is the **3rd step** in our optimization process.

The **4th, final step** is to choose, using the qualitative bounds of Proposition 4 (or more accurate ones), $\widetilde{n}_0(Z)$ for $Z<Z'$ so that the Z-electrons are not involved in any WB for $\xi<\xi_c$ preferably minimizing Z' (many possible choices).

Discussion and conclusions

Summarizing, the steps of our preliminary optimization process are:

1. finding the optimal value n_0^M of the plateau density n_0 maximizing $f(n_0)$;
2. finding one or a small number of pairs Z, Z' $(Z \gtrsim z_s \gtrsim Z')$ allowing solutions of (34); $Z - Z'$ is approximately given by (36);
3. finding $n_0' \equiv \widetilde{n}_0(Z')$ via (37) or more accurate computations;
4. adjusting $\widetilde{n}_0(Z)$ for $Z < Z'$ to avoid other WB for $\xi < \xi_c$.

Range of validity of the no-depletion approximation

One shows by self-consistency that a slowly modulated monochromatic pulse is not significantly affected by the plasma interaction in the intersection of stripes

$$0 \le ct - z \le l, \qquad 0 \le \frac{e^2 n_0 \lambda}{2mc^2}(ct+z) \ll 1. \tag{38}$$

By $\lambda \ll l$ and (14), this is much longer than l in the $(ct+z)$ direction: fine!

Conditions on the laser spot radius R

Under the above plane wave assumptions

$$\hat{\boldsymbol{\Delta}}^{\perp} \simeq \frac{-e\boldsymbol{\epsilon}^{\perp}}{k^2mc^2\hat{s}}, \qquad |\boldsymbol{\Delta}_M^{\perp}| \simeq \frac{e}{k^2mc^2}\max\left\{\frac{\epsilon}{\hat{s}}\right\} \leq \frac{a_0}{k}\left[h+\sqrt{h^2-\gamma^{\perp 2}}\right]. \qquad (39)$$

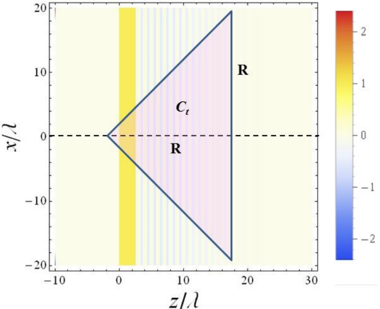
Actually, the *real* initial pulse is cylindrically symmetric around $\vec{z}$ and has a **finite** spot radius R, i.e. at $t=0$ $\quad \mathbf{E}=\boldsymbol{\epsilon}^{\perp}(-z)\,\chi(\rho)$, $\;\mathbf{B}=\mathbf{k}\times\mathbf{E}$, where $\rho^2=x^2+y^2$ and $\chi(\rho)\geq 0$ is 1 for $\rho\leq 1$ and $\chi(\rho)\to 0$ rapidly as $\rho\to\infty$. For the applicability of our results at least in some spacetime region it must be

$$R \gg |\boldsymbol{\Delta}_M^{\perp}|,\; l\,. \qquad (40)$$

Using causality arguments we can say that, as long as the pulse is not significantly depleted and its spot radius remains R:

1. The electron dynamics remains the same within the lightcone (with axis $\vec{z}$) trailing the pulse. This applies in particular to electrons self-injected in the PW.

2. If $R, \tilde{n}_0$ are small enough the *slingshot effect* (backward expulsion of energetic electrons from the plasma-vacuum interface) may occur [Fiore et al 2014-2016].



References

- G. Fiore, T. Akhter, S. De Nicola, R. Fedele, D. Jovanović, *On the impact of short laser pulses on cold diluted plasmas*, forthcoming paper
- G. Fiore, M. De Angelis, R. Fedele, G. Guerriero, D. Jovanović, Mathematics **10** (2022), 2622.
- G. Fiore, *Time-dependent harmonic oscillator revisited*, arXiv:2205.01781
- R. W. Assmann, et al., Eur. Phys. J.: Spec. Top., **229** (2020), 3675-4284.
- G. Fiore, P. Catelan, Nucl. Instr. Meth. Phys. Res. **A909** (2018), 41-45.
- G. Fiore, J. Phys. A: Math. Theor. **47** (2014), 225501.
- G. Fiore, R. Fedele, U. de Angelis, Phys. Plasmas **21** (2014), 113105.
- G. Fiore, S. De Nicola, Phys Rev. Acc. Beams **19** (2016), 071302 (15pp).
- G. Fiore, S. De Nicola, Nucl. Instr. Meth. Phys. Res. **A829** (2016), 104.
- G. Fiore, Ricerche Mat. **65** (2016), 491-503.
- G. Fiore, J. Phys. A: Math. Theor. **51** (2018), 085203.
- G. Fiore, P.Catelan, Ricerche Mat. **68** (2019), pp 341-357.