Sector Approach to Parton Showers

with Andrew Larkoski
and
Kassahun Betre

M. E. Peskin
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There is much interest now in rethinking the basic elements of QCD for simulation of LHC physics.

The most important issues are

Precision QCD: incorporating data from NLO, NNLO calcuations into simulations

Matching: building simulations that are correct with respect to LO QCD for large numbers of jets in the final state

Color coherence: building simulations that have the correct color structure in soft emission regions

In this talk, I would like to discuss some ideas that might be useful in creating more accurate parton showers toward these goals.

In this talk, I will discuss:

spin-dependent splitting

sectorized shower generation

antenna showers

I apologize that I will mainly discuss concepts that are not yet realized in code.

Other groups that are pursuing improved parton showers are

VINCIA (Giele, Kosower, Skands)
MENLOPS (Hoeche, Krauss, Siegert, Schonherr)
GenEvA (Bauer, Tackmann, Thaler)

Spin-dependent effects in Altarelli-Parisi splitting are large and generate momentum correlations within a shower.

	$++$	$-+$	$+-$	$--$
$g_+ \to gg :$	$1/z(1-z)$	$(1-z)^3/z$	$z^3/(1-z)$	0

Typically, parton showers sum over parton helicity at each stage. However, it is straightforward instead to sample helicity, that is, to generate specific helicity states in the Monte Carlo. The approximation made is that interference effects between different intermediate helicity states are neglected.

The process gives an event with labeled final helicities.

For massless partons, there is no interference between final helicity states. QCD cross sections are more easily obtained for definite final helicity than for helicity sums. Thus, we hope, this approach will facilitate QCD/parton shower matching.

Sector generation of parton showers attempts to cover phase space more systematically.

A parton shower is a good approximation for collinear emission. The approximation becomes inexact and needs correction for wide-angle emission. However, a parton shower might not generate events in all wide-angle regions, or might generate these regions multiple times starting from different collinear limits.

For matching, it is simplest to generate each point of phase space once and only once. The matching is done multiplicatively, by reweighting with the exact matrix element.

$$W = |\mathcal{M}(\{h_i\})_{LO\ QCD})|^2 / |\mathcal{M}(\{h_i\})|^2_{shower}$$

(GenEvA generates all regions of phase space from all collinear regions, with probability weighting.)

It is easiest to control the coverage of phase space by imposing strict virtuality ordering for all emissions

$$|s_{bc}| < |s_{ac}| < |s_{ab}| < \cdots$$

and summing over orderings (or, choosing orderings as part of the Monte Carlo generation).

Parton shower algorithms must include provision for recoil to conserve momentum. For example, PYTHIA associates each $1 \to 2$ splittling with a third particle that carries away the excess momentum.

In a sector approach, the recoil is minimized, hopefully leading to a better approximation to the kinematics.

There is a price in efficiency; emissions violating the ordering condition must be rejected.

Both of these ideas are naturally implemented in the context of an antenna shower, a parton shower in which the basic move is a $2 \to 3$ rather than a $1 \to 2$ splitting.

Antenna showers also provide an implementation of color flow in the event. I will discuss this in a moment.

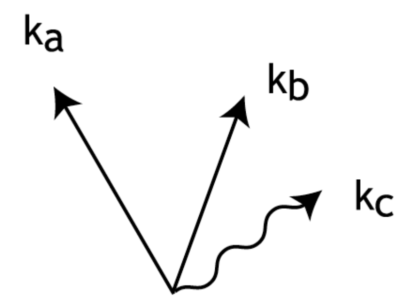
In QCD, it is much easier to generate color-ordered amplitudes than fully color-summed amplitudes. Matching with color-ordered amplitudes gives a parton shower corrected to leading order in $1/N_c^2$. Most likely, this is a small additional approximation to the approximation of using LO amplitudes.

So, in the rest of this talk, I will work within the leading large N_c approximation only.

Color coherence is usually modeled in parton showers using angular ordering:

Classic work of Marchesini and Webber, Mueller

$$\int \frac{d^3k_c}{(2\pi)^3 2k_c} \frac{2k_a \cdot k_b}{2k_a \cdot k_c 2k_c \cdot k_b} = 0$$



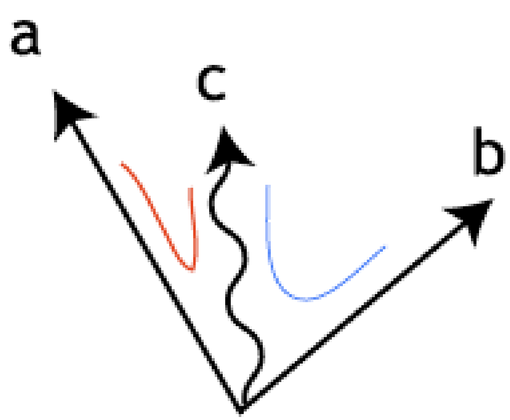
when c is radiated outside the cone of a,b.

HERWIG includes this effect by angular-ordering of its parton shower.

PYTHIA has a pT-ordered shower but vetos emissions that violate angular ordering.

In the late 1980's, members of the Lund group (Gustafson, Petterson, Lonnblad, Andersson) sought to do better by building a parton shower based intrinsically on radiation from QCD dipoles-- ARIADNE.

usual Altarelli-Parisi 1 → 2 splitting:



$$N_c \frac{\alpha_s}{4\pi} \int \frac{dp_T^2}{p_T^2} \int dz_a \mathcal{P}(z)$$

ARIADNE ('antenna') splitting:

$$N_c \frac{\alpha_s}{4\pi} \int dz_a dz_b \mathcal{S}(z_a, z_b, z_c)$$

I use a normalization appropriate to large Nc QCD.

What should we use for the $\mathcal{S}(z_a, z_b, z_c)$?

ARIADNE group:

$$q\bar{q} \longrightarrow qg\bar{q} \quad : \quad \frac{z_a^2+z_b^2}{(1-z_a)(1-z_b)}$$

$$qg \longrightarrow qgg \quad : \quad \frac{z_a^2+z_b^3}{(1-z_a)(1-z_b)}$$

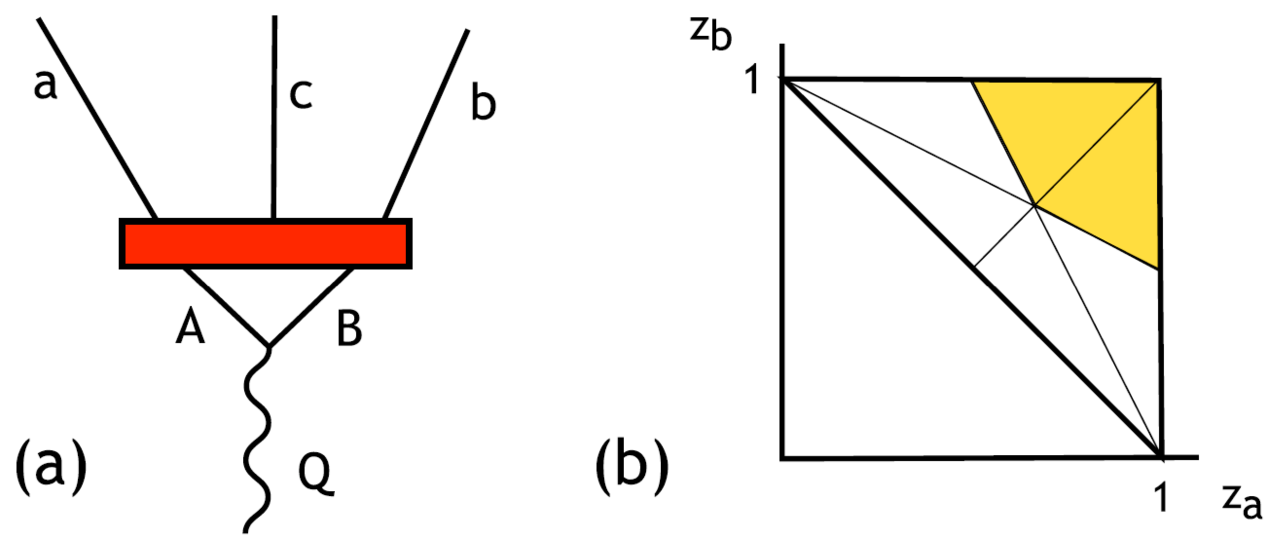
$$gg \longrightarrow ggg \quad : \quad \frac{z_a^3+z_b^3}{(1-z_a)(1-z_b)}$$

Gehrmann-de Ritter, Gehrmann, Glover:

different proposal emerging from 'antenna subtraction'

Altarelli-Parisi gave helicity-dependent splitting functions.
What are these for $\mathcal{S}(z_a, z_b, z_c)$?

Let's first systematically compute the $\mathcal{S}(z_a, z_b, z_c)$ for final-state radiation (FF kinematics):



$$z_a = \frac{2Q \cdot k_a}{s_{AB}} \ , \quad z_b = \frac{2Q \cdot k_b}{s_{AB}} \ , \quad z_c = \frac{2Q \cdot k_c}{s_{AB}}$$

$$y_{ab} = \frac{s_{ab}}{s_{AB}} \ , \quad y_{ac} = \frac{s_{ac}}{s_{AB}} \ , \quad y_{bc} = \frac{s_{bc}}{s_{AB}}$$

$$y_{ab} = (1 - z_c) \ , \quad y_{ac} = (1 - z_b) \ , \quad y_{bc} = (1 - z_a)$$

$$y_{ab} + y_{ac} + y_{bc} = 1 \ , \quad z_a + z_b + z_c = 2$$

We suggest the form for $\mathcal{S}(z_a, z_b, z_c)$:

$$\mathcal{S}(z_a, z_b, z_c) = \frac{\mathcal{N}(z_a, z_b, z_c)}{y_{ab} y_{ac} y_{bc}}$$

where $\mathcal{N}(z_a, z_b, z_c)$ is a polynomial. These functions must have the correct soft limit $z_a, z_b \to 1$:

$$\mathcal{S}(z_a, z_b, z_c) \to \frac{1}{y_{ac} y_{bc}} \delta_{A,a} \delta_{B,b}$$

and the correct collinear limits, e.g., $z_b \to 1$

$$\mathcal{S}(z_a, z_b, z_c) \to \frac{1}{y_{ac}} \mathcal{P}_{A \to ac}(z_c)$$

Where both collinear limits are nonvanishing, these requirements actually fix the numerator. In other cases, we must do a computation.

Here are the results:

	$+++$	$++-$	$+-+$	$-++$	$--+$	$-+-$	$+--$	$---$
$g_+g_+ \to ggg$	1	y_{ac}^4	y_{ab}^4	y_{bc}^4	0	0	0	0
$g_-g_+ \to ggg$	0	0	y_{bc}^4	z_a^4	z_b^4	y_{ac}^4	0	0
$g_+g_+ \to \overline{q}qg$	-	-	$y_{ab}^3 y_{bc}$	$y_{ab} y_{bc}^3$	-	0	0	-
$g_-g_+ \to \overline{q}qg$	-	-	$y_{ab} y_{bc}^3$	$z_a^2 z_b^2 y_{ab} y_{bc}$	-	0	0	-
$q_-\overline{q}_+ \to qg\overline{q}$	-	-	-	$y_{ab} z_b^2$	$y_{ab} z_a^2$	-		-
$q_-\overline{q}_- \to qg\overline{q}$	-	-	-	-	-	y_{ab}^3	-	y_{ab}
$q_-g_- \to qgg$	-	-	-	0	y_{ac}^4	$y_{ab}^3 z_b$	-	z_a
$q_-g_+ \to qgg$	-	-	-	z_a^3	$y_{ab} z_b^3$	y_{ac}^4	-	0
$q_-g_- \to q\overline{q}q$	-	-	-	-	$y_{ab} y_{ac}^3$	$y_{ab}^2 y_{ac} z_b$	-	-
$q_-g_+ \to q\overline{q}q$	-	-	-	-	$z_a y_{ab} y_{ac} z_b^2$	$y_{ab} y_{ac}^3$	-	-

Please note that, away from the soft and collinear limits, this splitting function has no universal definition. So many choices are possible. We think that these are the simplest ones.

Some of the expressions with quarks get complicated, but the pure gluon splitting amplitudes -- the core of any parton shower -- are very simple

	$+++$	$++-$	$+-+$	$-++$
$g_+g_+ \to ggg$	1	y_{ac}^4	y_{ab}^4	y_{bc}^4

	$+-+$	$-++$	$--+$	$-+-$
$g_-g_+ \to ggg$	y_{bc}^4	z_a^4	z_b^4	y_{ac}^4

I have written these for the case of final-final emissions, but the same expressions apply to all emission regions.

I have explained the derivation in the language of final-state emission, but actually this derivation is correct in any kinematic regime. The antenna functions for initial-state emissions are obtained by analytic continuation of these formulae.

Our master expression for the spin-dependent splitting functions

$$\mathcal{S}(z_a, z_b, z_c) = Q^2 \left| \frac{\mathcal{M}(\mathcal{O} \to acb)}{\mathcal{M}(\mathcal{O} \to AB)} \right|^2$$

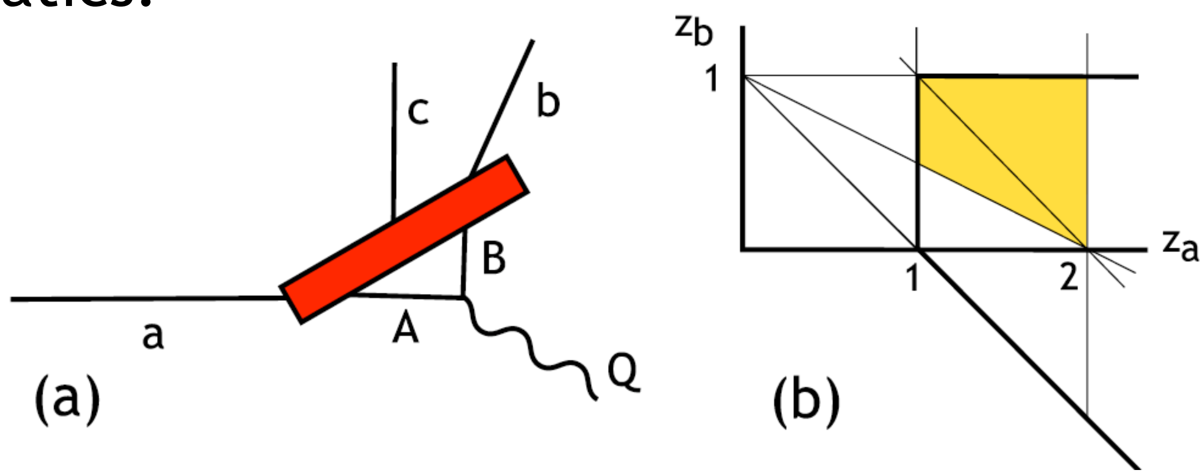
always continues to a positive result, at least in the regions where z_a, z_b, z_c are all positive.

Now we have the ingredients for an antenna shower representing final-state radiation. We still need an explicit implementation of the kinematics suitable for a computer program. I will discuss this in a moment. First, let's discuss how this formalism generalizes to initial-state radiation.

We must consider two new types of antennae: Initial-Final (IF) and Initial-Initial (II) radiators. We will consider these cases as continuations from the FF region, and use the same variables y and z.

Krauss and Winter have implemented these antennae in SHERPA using the ARIADNE splitting functions.

IF kinematics:



The basic cross section formula is:

$$\sigma(pX \to cb) = \int dx_a f(x_a) \; \frac{1}{\Phi_{aX}} \frac{1}{16\pi} \int d\cos\theta_* \; |\mathcal{M}(aX \to cb)|^2$$

Fixing Q in the transition from AB to acb, with a collinear with A, this formula can be rewritten as:

$$\sigma(pX \to cb) = \int \frac{dz_a}{z_a^2} \int dz_b \int dx_A x_A f(z_a x_A) \; \delta(x_A + Q^2/2P \cdot Q)$$
$$\cdot \frac{1}{\Phi_{AX}} \frac{1}{8\pi} |\mathcal{M}(aX \to cb)|^2 \; .$$

Now make the approximation that the matrix element factorizes. We find

$$\sigma(pX \to cb) \approx \int \frac{dz_a}{z_a^2} \int dz_b \int dx_A f(z_a x_A) \sigma(AX \to B) \cdot \frac{\alpha_s N_c}{4\pi} \mathcal{S}(z_a, z_c, z_b)$$

This can be shown to have the correct limits as c becomes collinear with a or b.

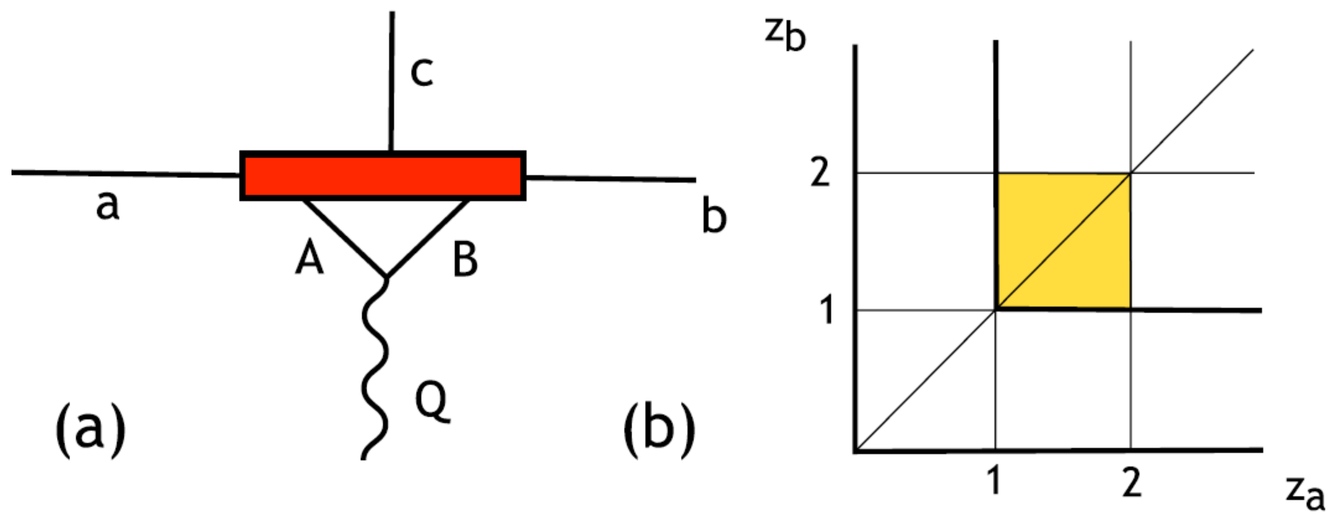
To implement an antenna shower, start from the AX→B process, and replace (AB) by a system (acb) with the same momentum transfer Q. The structure function must be evaluated at a higher longitudinal fraction. In the limit of c collinear with a, the relation goes to

$$x_a = x_A / w$$

where w is the energy fraction transferred to initial state radiation. In the limit in which c becomes collinear with b, we have

$$x_a = x_A$$

II kinematics:



The basic cross section formula is:

$$\sigma(pp \to cX) = \int dx_a \int dx_b f(x_a) f(x_b) \; \frac{1}{2s_{ab}} \frac{1}{16\pi} \int d\cos\theta_* \frac{2p_*}{\sqrt{s_{ab}}} \, |\mathcal{M}(ab \to cX)|^2$$

Fixing Q in the transition from (AB) to (acb), this formula can be rewritten as:

$$\sigma(pp \to cX) = \int \frac{dz_a}{z_a^2} \frac{dz_b}{z_b^2} \frac{1}{\mathcal{C}^4} \; \int dx_A dx_B \; f(z_a x_A \mathcal{C}) f(z_b x_B \mathcal{C}) \; x_B \delta(x_B - Q^2/x_A 2P_A \cdot P_B)$$
$$\cdot \frac{1}{s_{AB}} \frac{1}{8\pi} |\mathcal{M}(aX \to cb)|^2 \; .$$

Note that (AB) cannot be maintained collinear with (ab).

with $\mathcal{C}^2 = \frac{z_a + z_b - 1}{z_a z_b}$

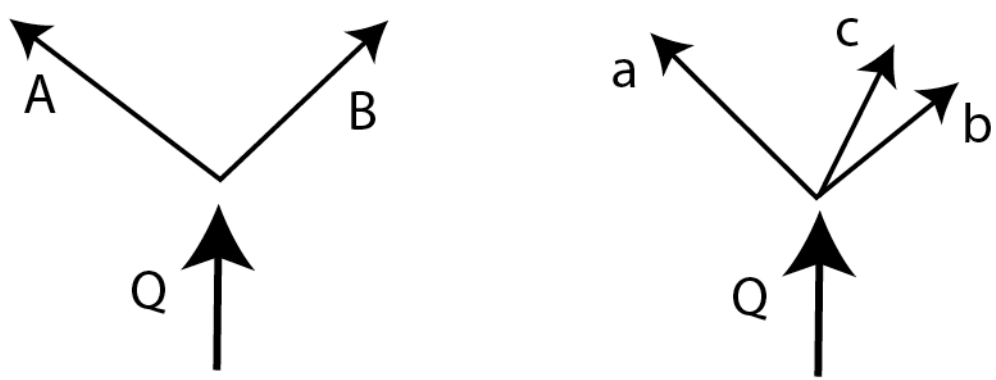
Making the approximation that the matrix element factorizes, we find

$$\sigma(pp \to cX) \approx \int \frac{dz_a}{z_a^2}\frac{dz_b}{z_b^2}\frac{1}{\mathcal{C}^4} \int dx_A dx_B\, f(z_a x_A \mathcal{C}) f(z_b x_B \mathcal{C}) \sigma(AB \to X) \cdot \frac{\alpha_s N_c}{4\pi}\, \mathcal{S}(z_a, z_c, z_b)$$

This expression correctly partitions the extra energy needed for initial-state radiation between a and b. It can be shown to give the correct answer in both collinear limits.

Next, we need the momentum transformation in each region.

In an antenna shower, the most straightforward way to add a parton is through antenna replacement (ARP) (VINCIA): keep the same total vector Q, replace AB by acb



In the FF region, it is easy to construct acb:

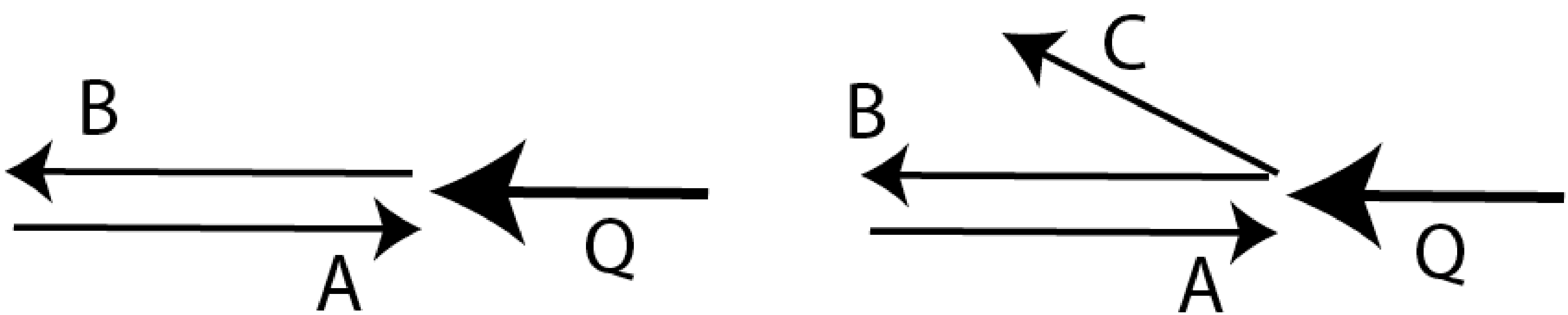
Boost so that A and B are back to back, then Q = (Q,0,0).
Add C, boost along c so that A+B+C = (Q',0,0).
Rescale so that the total is Q.
Rotate by ϕ.
Undo the first boost and insert acb in place of AB.

For $\quad A=(1,0,1) \qquad B=(1,0,-1)$

$$
\begin{aligned}
a &= \left(z_a \ , \ -(y_{ab}y_{bc})^{1/2}(1+\frac{y_{ac}-y_{bc}}{(1+y_{ab}^{1/2})^2}) \ , \ y_{ab}^{1/2}+\frac{y_{ac}-y_{bc}}{2}(1+\frac{y_{ac}-y_{bc}}{(1+y_{ab}^{1/2})^2})\right) \\
b &= \left(z_b \ , \ -(y_{ab}y_{bc})^{1/2}(1-\frac{y_{ac}-y_{bc}}{(1+y_{ab}^{1/2})^2}) \ , \ -y_{ab}^{1/2}+\frac{y_{ac}-y_{bc}}{2}(1-\frac{y_{ac}-y_{bc}}{(1+y_{ab}^{1/2})^2})\right) \\
c &= \left(z_c \ , \ 2(y_{ac}y_{bc})^{1/2} \ , \ -(y_{ac}-y_{bc})\right)
\end{aligned}
$$

This generalizes straightforwardly to the IF region.   We have vectors A in the initial beam direction and B in the final state.

Boost to the Breit frame and add C



To restore Q to its original direction, we need a Lorentz transformation that rotates B and C while preserving the direction of A.   This is not hard to find. There is a 3-parameter family of Lorentz transformations that leave A invariant; we can apply an appropriate one and then boost along the beam direction.

More specifically, in the coordinate system $(V^+, V_\perp, V^-)$ the transformation

$$\Lambda = \begin{pmatrix} b & 2ab & a^2 b \\ 0 & 1 & a \\ 0 & 0 & b^{-1} \end{pmatrix} \text{ with } \quad a = -\frac{(y_{ac} y_{bc})^{1/2}}{y_{ab} + y_{ac}} \qquad b = \frac{y_{ab} + y_{ac}}{\sqrt{y_{ab}}}$$

restores Q to its original value. We can then replace AB with acb. A property is that a is longer than A (backwards evolution)

For $A = (1, 0, 1) \qquad B = (1, 0, -1)$

$$\begin{aligned}
a &= (z_a, 0, z_a) \\
b &= \left(\frac{y_{ab} + y_{bc} y_{ac}}{z_a}, \; -2\frac{(y_{ab} y_{ac} y_{bc})^{1/2}}{z_a}, \; \frac{-y_{ab} + y_{ac} y_{bc}}{z_a}\right) \\
c &= \left(\frac{y_{ac} + y_{bc} y_{ab}}{z_a}, \; +2\frac{(y_{ab} y_{ac} y_{bc})^{1/2}}{z_a}, \; \frac{-y_{ac} + y_{ab} y_{bc}}{z_a}\right)
\end{aligned}$$

Finally, for the II region, there is no freedom except to do boosts along the beam axis. We can boost to remove the longitudinal component of C, but the transverse component of C must be balanced by a transverse boost of Q.

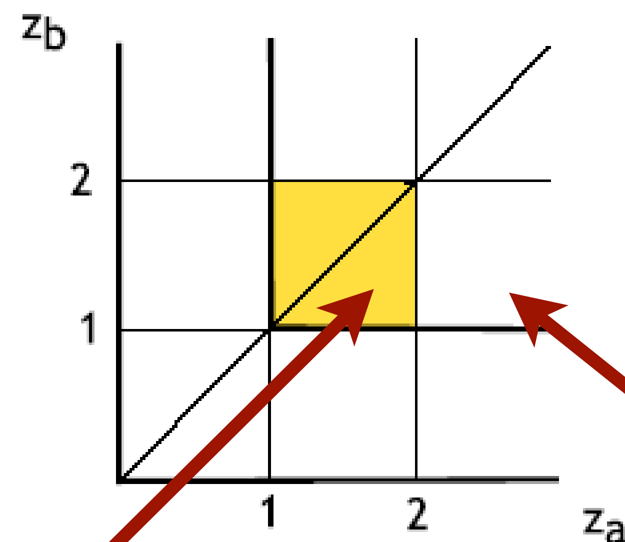
For $A = (1, 0, 1) \qquad B = (1, 0, -1)$

$$
\begin{aligned}
a &= (z_a - y_{ac})(\frac{y_{ab}}{y_{ab} + y_{ac}y_{bc}})^{1/2} \; (1, 0, 1) \\
b &= (z_b - y_{bc})(\frac{y_{ab}}{y_{ab} + y_{ac}y_{bc}})^{1/2} \; (1, 0, -1) \\
c &= (\frac{y_{ab}^2 - 1 + (y_{bc} - y_{ac})^2}{2(y_{ab}(y_{ab} + y_{ac}y_{bc}))^{1/2}} \; , \; 2(\frac{y_{ac}y_{bc}}{y_{ab}})^{1/2} \; , \; \frac{y_{ab}(y_{bc} - y_{ac})}{(y_{ab}(y_{ab} + y_{ac}y_{bc}))^{1/2}})
\end{aligned}
$$

and a boost of Q by

$$
\beta = -(\frac{y_{ac}y_{bc}}{y_{ab} + y_{ac}y_{bc}})^{1/2}
$$

There is one more important subtlety with the sector treatment of initial state radiation. Consider again the z plane, thinking, for definiteness, about the Drell-Yan final state $q\bar{q} \to W gg$



In this region, we have emission of a gluon correcting
$q\bar{q} \to W g$

In this region, we have emission of a W correcting
$q\bar{q} \to gg$

It is not so clear how to generate the second region correctly without generating all LHC QCD reactions.

So far, all of these results are for all massless particles.

However, at the LHC we will have many energetic top quarks, and it will be interesting to study gluon radiation from top quarks.

How do we do this in an antenna framework ?

Massive Altarelli-Parisi splitting function $t \to gt$
(Catani-Dittmaier-Trocsanyi-Seymour)

$$P(z) = \frac{4}{3}\left[\frac{1+(1-z)^2}{z} - \frac{m^2}{k_t \cdot k_g}\right]$$

To understand this, it is interesting to compare to the spin-dependent gluon emission amplitudes:

$$\begin{aligned}
i\mathcal{M}(t_L \to g_L t_L) &= \sqrt{2} i g \frac{p_T}{z(1-z)^{1/2}} \cdot 1 \\
i\mathcal{M}(t_L \to g_R t_L) &= \sqrt{2} i g \frac{p_T}{z(1-z)^{1/2}} \cdot (1-z) \\
i\mathcal{M}(t_L \to g_L t_R) &= \sqrt{2} i g \frac{m}{z(1-z)^{1/2}} \cdot z^2 \\
i\mathcal{M}(t_L \to g_R t_R) &= \sqrt{2} i g \frac{m}{z(1-z)^{1/2}} \cdot 0
\end{aligned}$$

Square and integrate with phase space and the massive denominator $1/(P^2 - m^2)^2$, with

$$P^2 - m^2 = \frac{p_T^2 + z^2 m^2}{z(1-z)}$$

and make use of $\quad 2g^2 \int \frac{d^3p}{(2\pi)^3 2E} = \frac{\alpha_s}{2\pi} \int \frac{dz}{z} \int dp_T^2$

Then we find for the emission probability,

$$\frac{\alpha_s}{2\pi} \int dz \int \frac{dp_T^2}{[p_T^2 + z^2 m^2]^2} \left(\frac{p_T^2 (1 + (1-z)^2)}{z} + \frac{m^2 z^4}{z} \right)$$

$$= \frac{\alpha_s}{2\pi} \int dz \int \frac{dp_T^2}{[p_T^2 + z^2 m^2]} \left(\frac{1 + (1-z)^2}{z} - \frac{2m^2}{p_T^2 + z^2 m^2} \right)$$

The first of these formulae is more illuminating. There is an extra term specifically from helicity-flip emission, and it shows a "dead cone" for soft radiation from slow top quarks.

Can we find analogous formulae for antennae ?

Our strategy was to consider the antenna as generated by a local operator. Then the antenna splitting function is

$$Q^2 \left| \frac{\mathcal{M}(\mathcal{O} \to 3)}{\mathcal{M}(\mathcal{O} \to 2)} \right|^2$$

to be integrated over 3-particle phase space. This formula can still be used in the massive case. The FF splitting probability is

$$\int d\,\mathrm{Prob} = N_c \frac{\alpha_s}{4\pi} (\frac{Q}{2K}) \int dz_a dz_b \, \mathcal{S}(z_a, z_b, z_c)$$

The computation of heavy quark amplitudes is easier if we use spinor products. For heavy quarks, we use the Schwinn-Weinzierl representation of the massive spinors,

$$\langle t_L| = \frac{[q(k+m)}{[qk^\flat]} \qquad \langle t_R| = \frac{\langle q(k+m)}{\langle qk^\flat\rangle}$$

with

$$k^\flat = k - \frac{m^2}{2k\cdot q}q$$

q is a massless reference vector. It is most convenient to choose q to be the massless vector in the backwards direction to k . We call this $k^\sharp$. Then

$$k = (E,0,0,k)\ , \quad k^\flat = \frac{E+k}{2}(1,0,0,1)\ , \quad k^\sharp = \frac{E+k}{2}(1,0,0,-1)$$

With this definition, the above spinors correspond to the standard helicity states.

Analyze the simplest case of a $t\overline{q}$ dipole, where q is a light quark. This has spin 0 and spin 1 cases. The spin 0 case is quite simple.

The chiral local operator generating this dipole is $t_L\overline{q}_L$. The 2-body matrix elements $\mathcal{O} \to t(A)\overline{q}(B)$ are

$$\begin{aligned} i\mathcal{M}(\mathcal{O} \to t_L\overline{q}_L) &= \langle A^\flat B\rangle \\ i\mathcal{M}(\mathcal{O} \to t_R\overline{q}_L) &= m\langle qB\rangle/\langle qA^\flat\rangle \end{aligned}$$

Taking $q = A^\sharp$, and recognizing that $A^\sharp$ is parallel to B, we have

$$\langle A^\sharp B\rangle = 0$$

Then we have the standard helicity rule that a spin zero state has only $t_L\overline{q}_L$ and no $t_R\overline{q}_L$.

The matrix elements of the operator to 3-particle states are (using a general reference vector q):

$$\mathcal{M}(Q_L g_L \overline{q}_L) = -\frac{1}{[qc]}\left\{\frac{\langle c a^\flat\rangle[qQb\rangle}{s_{ac}-m^2} + \frac{[qQa^\flat\rangle}{[bc]}\right\}$$

$$\mathcal{M}(Q_L g_R \overline{q}_L) = -\frac{\langle a^\flat b\rangle[cQb\rangle}{\langle bc\rangle(s_{ac}-m^2)}$$

$$\mathcal{M}(Q_R g_L \overline{q}_L) = -\frac{m}{[a^\flat c]\langle q a^\flat\rangle}\left\{\frac{\langle cq\rangle[a^\flat Qb\rangle}{s_{ac}-m^2} + \frac{[a^\flat Qq\rangle}{[bc]}\right\}$$

$$\mathcal{M}(Q_R g_R \overline{q}_L) = -\frac{m\langle qb\rangle[cQb\rangle}{\langle bc\rangle\langle q a^\flat\rangle(s_{ac}-m^2)}$$

Choosing $q = a^\sharp$ and noting that $[a^\sharp Q a^\flat\rangle = 0$, these simplify to single-term expressions.

We believe it is easiest to keep these expressions in terms of spinor products. We are trying to implement them by generating massless phase space and then rescaling (as in RAMBO) to the massive case. Then $a^\flat$ and $a^\sharp$ are parallel and opposite to the original massless vectors.

Massive fermion + quark (spin 0)

$$\mathcal{S}(Q_L g_L \overline{q}_L) = \frac{Q^2}{Q^2 - m^2}\left|\frac{\langle a^\flat c\rangle[a^\sharp Q b\rangle}{[a^\sharp c][cac\rangle}\right|^2$$

$$\mathcal{S}(Q_L g_R \overline{q}_L) = \frac{Q^2}{Q^2 - m^2}\left|\frac{\langle a^\flat b\rangle[cQb\rangle}{\langle bc\rangle[cac\rangle}\right|^2$$

$$\mathcal{S}(Q_R g_L \overline{q}_L) = \frac{m^2}{Q^2 - m^2}\left|\frac{\langle a^\sharp c\rangle[a^\flat Q b\rangle}{[a^\flat c][cac\rangle}\right|^2$$

$$\mathcal{S}(Q_R g_R \overline{q}_L) = \frac{m^2}{Q^2 - m^2}\left|\frac{\langle a^\sharp b\rangle[cQb\rangle}{\langle bc\rangle[cac\rangle}\right|^2$$

Massive fermion+ quark (spin1)

$$\mathcal{S}(Q_L g_R \overline{q}_R) = \frac{Q^2}{(Q^2 - m^2)^2}\left|\frac{\langle a^\flat B\rangle[A^\flat(b+c)ac]}{\langle cac]\langle bc\rangle}\right|^2$$

$$\mathcal{S}(Q_L g_L \overline{q}_R) = \frac{1}{(Q^2 - m^2)^2}\left|\frac{[A^\flat b]}{\langle cac][bc]}\left\{[a^\sharp ac\rangle[bQB\rangle + m^2\langle cB\rangle[a^\sharp b]\right\}\right|^2$$

$$\mathcal{S}(Q_R g_R \overline{q}_R) = \frac{m^2}{(Q^2 - m^2)^2}\left|\frac{\langle a^\sharp B\rangle[A^\flat(b+c)ac]}{\langle cac]\langle bc\rangle}\right|^2$$

$$\mathcal{S}(Q_R g_L \overline{q}_R) = \frac{m^2}{(Q^2 - m^2)^2}\left|\frac{[A^\flat b]}{\langle cac]\langle bc\rangle}\left\{\langle a^\sharp B\rangle\langle cab] + \langle a^\sharp c\rangle\langle Bcb]\right\}\right|^2$$

We hope that these methods will eventually be realized in an antenna-based, color-coherent parton shower for massless and massive particles.